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A Weak Galerkin Method with RT Elements for a Stochastic Parabolic Differential Equation

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Abstract. A weak Galerkin finite element method with Raviart-Thomas elements for a linear stochastic parabolic partial differential equation with space-time additive noise is studied and optimal strong convergence error estimates in L^2-norm are obtained.

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Key words: Weak Galerkin method, weak gradient, stochastic PDE, standard counterparts, Raviart-Thomas element.

1. Introduction

Stochastic partial differential equations (SPDEs) are used to describe various stochastic phenomena in natural sciences [14, 17, 20, 21, 26, 33]. In particular, they serve as mathematical models in physics [7], chemistry [31], biology [3], etc. SPDEs can be written in the form

$$du = (Au + F(u))dt + B(u)dW(t), \quad u(0) = u_0,$$

where H is a Hilbert space, $u(t)$ an H-valued stochastic process, A a linear, self-adjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined on $\mathscr{D} \subset H$. Besides, F and B are nonlinear operators in H, $W(t)$ is an H-valued Q-Wiener process defined on a filtrated probability space $(\Omega, \mathscr{F}, P)$ and $u_0 \in H$.

Stochastic partial differential equations are more complex and lack many properties of regular deterministic operators — cf. Refs. [10, 11, 13, 27, 32]. In this work, we explore a weak Galerkin finite element method with Raviart-Thomas (RT) elements for the linear stochastic parabolic partial differential equation

$$\begin{aligned} &du + Audt = dW && \text{in } \mathscr{D}, && 0 \le t \le T, \\ &u = 0 && \text{on } \partial\mathscr{D}, && 0 \le t \le T, \\ &u(0) = u_0 && \text{in } \mathscr{D}. && \end{aligned} \tag{1.1}$$

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The problem (1.1) has been treated by various numerical methods, the main of which can be divided into two groups: numerical difference methods [12, 15, 16] and the ones using finite dimensional subspaces. There are different approaches to the approximation of white noise in (1.1) but their numerical implementation requires substantial efforts. Thus Allen *et al.* [2] use piecewise constant functions in time and space to approximate white noise. Shardlow [29] applies a spectral method and employs truncated Fourier series for white noise approximations. Benth and Gjerde [4] exploit chaos expansions to approximate the Eq. (1.1). Yan [37,38] applies a standard finite element method to (1.1) with the operator $A=-\Delta$ and obtains optimal error estimates in L^2- and H^{-1}-norms. Larsson and Mesforush [18] approximate the Eq. (1.1) with $A=-\Delta^2$ by mixed finite element methods and establish error estimates related to the smoothness of the solution process. In contrast to spatial discretisation in [18], Chai *et al.* [5] discretise spatial variables and an infinite dimensional (cylindrical) Wiener process by Argyris finite elements and truncated stochastic series spanned by the spectral basis of the covariance operator, respectively. Raphael and Yue [28] investigated the semilinear form of the Eq. (1.1) and used a novel fully discrete numerical approximations, which combined standard Galerkin finite element method and a randomized Runge-Kutta scheme. They also proved the convergence of the method to a mild solution in L^p-norm for $p \in [2, \infty)$.

The weak Galerkin (WG) finite element method is introduced by Wang and Ye [22,34] in order to approximate second order elliptic problems. The finite element space consists of completely discontinuous functions defined in the interior and on the boundary of each partition element. The advantage of the method is that the corresponding base functions can be chosen as low order polynomials. More precisely, one can use the piecewise constant functions to construct a finite element space, so that the freedom of the linear equations derived can be very small. On the other hand, the weak gradient of the finite element functions implies that they are restricted to Raviart-Thomas and Brezzi-Douglas-Marini elements [1, 8] thus limiting the flexibility of applications. To overcome this problem, a stabilisation term is introduced to penalise the difference between standard differential operators and their weak versions [24]. Such weak Galerkin methods find numerous applications in various elliptic problems [23, 35], Helmholtz problem [25], Stokes equation [36], eigenvalue problems [39], parabolic problems [6, 40–42] and stochastic partial differential equations [19, 43].

Solving the Eq. (1.1) is a very challenging problem and computational cost is high. In order to reduce the computational complexity, here we employ the original weal Galerkin method without the stabiliser to discretise the spatial variables in the Eq. (1.1). We establish the optimal approximation order in L^2-norm. However, in contrast to the work of Chai *et al.* [6], a special structure of the weak finite element functions prevents us from obtaining H^1- and H^{-1}-norm estimates.

The rest of this paper is organised as follows. In Section 2, we introduce necessary functional spaces (of the Sobolev type) and describe a stochastic partial differential equation. Section 3 presents a weak Galerkin method with RT elements and for a stochastic partial differential equation. In Section 4 we study the errors of the semi-discrete weak Galerkin approximations and show that the method has optimal error estimates in L^2-norm.