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The Liouville Type Theorem for a System of Nonlinear Integral Equations on Exterior Domain

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Abstract. In this paper we are concerned with a system of nonlinear integral equations on the exterior domain under the suitable boundary conditions. Through the method of moving planes in integral forms which has some innovative ideas we obtain that the exterior domain is radial symmetry and a pair of positive solutions of the system is radial symmetry and monotone non-decreasing. Consequently, we can obtain the corresponding Liouville type theorem about the solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded connected domain with $\partial\Omega \in C^1$ and let $\Omega_1 = \mathbb{R}^n \setminus \overline{\Omega}$. We are concerned with the following system of nonlinear integral equations

$$\begin{cases} u(x) = \displaystyle\int_{\Omega_1} \frac{1}{|x-y|^{n-\alpha}} f(v(y)) g(\nabla v(y))\, \mathrm{d}y, & \text{in } \Omega_1, \\ v(x) = \displaystyle\int_{\Omega_1} \frac{1}{|x-y|^{n-\beta}} f(u(y)) g(\nabla u(y))\, \mathrm{d}y, & \text{in } \Omega_1, \end{cases} \tag{1.1}$$

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under boundary conditions

$$u=0 \text{ and } v=0, \text{ on } \partial\Omega, \tag{1.2}$$

where $1<\alpha,\ \beta<n$ and $\alpha+\beta<n$ for constants $\alpha,\ \beta$, with respect to the functions f and g we assume that

(fg_1) f and g are both positive functions;

(fg_2) $f(u)g(\nabla u)(x)\in L^1(\Omega_1)\cap L^{\frac{p}{p-1}}(\Omega_1)\cap L^{\frac{q}{q-1}}(\Omega_1)$, where $1<p<\min\left\{\frac{n}{n-\alpha},\frac{n}{n-\beta}\right\}$, $1<q<\min\left\{\frac{n}{n-\alpha+1},\frac{n}{n-\beta+1}\right\}$, $u(x)\in L^r(\Omega_1)\cap L^s(\Omega_1)$, $r>\frac{n}{n-\alpha}$ and $s>\frac{n}{n-\beta}$;

(fg_3) $f(u)$ and $g(w)$ are continuous in u and w, respectively;

(fg_4) $f^{'}(u)$ is nonnegative and $f^{'}(u)(x)\in L^t(\Omega_1)\cap L^{t_1}(\Omega_1)$, where $t=\frac{nrs}{ns+\alpha rs-nr}>1$, $t_1=\frac{nrs}{nr+\beta rs-ns}>1$, $u(x)\in L^r(\Omega_1)\cap L^s(\Omega_1)$, $r>\frac{n}{n-\alpha}$ and $s>\frac{n}{n-\beta}$;

(fg_5) $g(w(x_1,x_2,\dots,x_n))$ is non-decreasing in $|x_i|$, where $i\in[1,n]$ is a positive integer.

The system (1.1) implies the following system

$$\begin{cases}(-\Delta)^{\frac{\alpha}{2}}u=f(v)g(\nabla v), & \text{in } \Omega_1,\\ (-\Delta)^{\frac{\beta}{2}}v=f(u)g(\nabla u), & \text{in } \Omega_1.\end{cases} \tag{1.3}$$

Let $u=v$ and $\alpha=\beta$, the system (1.3) can be rewritten as the following nonlinear partial differential equation (PDE) on exterior domain

$$(-\Delta)^{\frac{\alpha}{2}}u=f(u)g(\nabla u), \quad \text{in } \Omega_1.$$

Let $\alpha=2$, we can get the following classical Laplacian equation

$$-\Delta u=f(u)g(\nabla u). \tag{1.4}$$

In [1], Serrin introduced that the equation $-\Delta u=f(u,|\nabla u|)$ which is the general form of (1.4) has the strong significance in physics. For instance, when we consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω, we can obtain it. Serrin studied the following problem

$$\begin{cases}-\Delta u=f(u,|\nabla u|), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\\ \frac{\partial u}{\partial\gamma}=C_1\le 0, & \text{on } \partial\Omega,\end{cases} \tag{1.5}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open connected domain. Serrin obtained that Ω is a ball and the positive solution of (1.5) is radially symmetric.