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The Invariant Rings of the Generalized Transvection Groups in the Modular Case

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Abstract: In this paper, first we investigate the invariant rings of the finite groups $G \leq \mathrm{GL}(n, F_q)$ generated by i-transvections and i-reflections with given invariant subspaces H over a finite field F_q in the modular case. Then we are concerned with general groups $G_i(\omega)$ and $G_i(\omega)^t$ named generalized transvection groups where ω is a k-th root of unity. By constructing quotient group and tensor, we calculate their invariant rings. In the end, we determine the properties of Cohen-Macaulay, Gorenstein, complete intersection, polynomial and Poincare series of these rings.

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1 Introduction

Let F_q be a finite field, where $q = p^\nu$, $\nu \in \mathbf{Z}_+$. Suppose that $x_1, \cdots, x_n \in V = F_q^n$ form a basis and $z_1, \cdots, z_n \in V^*$ form the dual basis to $\{x_1, \cdots, x_n\}$. We denote by $F_q[V]$ the graded ring of polynomial functions on V, which is defined to be the symmetric algebra on V^*. Hence $F_q[V] = F_q[z_1, \cdots, z_n]$. If G is a finite group, and $\rho\colon G \hookrightarrow \mathrm{GL}(n, F_q)$ is a representation of G over F_q, then, via ρ, G acts on the left of the vector space $V = F_q^n$. The invariant ring (see [1], Page 4), denoted by $F_q[V]^G$, is

$$F_q[V]^G = \{f \in F_q[V] \mid g \cdot f = f, \ \forall g \in G\}.$$

This is a graded subring of $F_q[V]$.

In this paper, we are mainly concerned with the invariant rings of the groups G generated

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by elements with the codimension i and some related properties over a finite field F_q in the modular case. In this case, the order of G is divided by the characteristic of the field F_q.

A modern algorithm for the construction of invariant ring of group G generated by elements with the codimension 1 can be found in [1]. This problem is originally considered by Landweber and Stong[2] in connecting with their study of the depth of invariant ring. Years earlier Nakajima[3] studies the dual representations. And there is another way to obtain these results (see [4]). Much of the argument is devoted to showing that the invariant rings are polynomial.

For the group G_i^+ defined in Definition 1.3, Neusel *et al.*[5]–[6] construct the invariant ring $F_q[V]^{G_i^+}$. In [5], Neusel and Smith adopt a method associated to configurations of hyperplanes. In [6], Nakajima determines finite irreducible subgroups G of $\mathrm{GL}(V)$ such that $F_q[V]^G$ are polynomial in the modular case.

A plan of the paper follows. In the remainder of this section, we illustrate the terminology used in this paper. In Section 2, we demonstrate the invariant rings of the groups G_i^+ and G_i generated by i-transvections and i-reflections, respectively. Constructing quotient group and tensor is the key ingredient in the approach applied in this part. In Section 3, involving a k-th root of unity ω we define a general group $G_i(\omega)$ named generalized transvection group. Then we investigate the invariant ring of $G_i(\omega)$ and its properties of Cohen-Macaulay, Gorenstein, complete intersection, polynomial and Poincare series. In the last section, we consider another generalized transvection group $G_i(\omega)^t$ which is the transpose of the group $G_i(\omega)$ and its properties.

We begin with a short review of some basic definitions concerning invariant and pseudo-reflection as a preliminary to introduce i-transvection and i-reflection we need in this paper. We adopt the definitions from [5] and [7].

Definition 1.1[7] *Given an element* $\boldsymbol{T} \in \mathrm{GL}(n, F_q)$. *Denote the dimension of the subspace* $\mathrm{Im}(\boldsymbol{I}-\boldsymbol{T}) \subset V = F_q^n$ *over* F_q *by* $\mathrm{Res}(\boldsymbol{T})$. *So the dimension of the subspace* $\ker(\boldsymbol{I}-\boldsymbol{T})$ *over* F_q *is equal to* $(n-\mathrm{Res}(\boldsymbol{T}))$.

In a finite group $G \subseteq \mathrm{GL}(n, F_q)$, a pseudo-reflection (see [5]) $\boldsymbol{T} \in G$ satisfies $\dim_{F_q}(\mathrm{Im}(\boldsymbol{I}-\boldsymbol{T})) = 1$, i.e., $\mathrm{Res}(\boldsymbol{T}) = 1$. A pseudo-reflection $\boldsymbol{T} \neq \boldsymbol{I}$ is called a transvection if $\boldsymbol{T}|_{(\boldsymbol{I}-\boldsymbol{T})V} = \boldsymbol{I}$, and a reflection if $\boldsymbol{T}|_{(\boldsymbol{I}-\boldsymbol{T})V} = -\boldsymbol{I}$. Similarly, we define i-transvection and i-reflection.

Definition 1.2 *Denote the floor of a number* $t \in \mathbf{Q}$ *by* $[t]$. *Let* $\boldsymbol{T} \in \mathrm{GL}(n, F_q)$ *satisfy* $\mathrm{Res}(\boldsymbol{T}) = i$, *where* $1 \le i \le \left[\frac{n}{2}\right]$. *Then* $\boldsymbol{T}$ *is called an* i*-transvection if* $\boldsymbol{T}|_{(\boldsymbol{I}-\boldsymbol{T})V} = \boldsymbol{I}$, *and an* i*-reflection if* $\boldsymbol{T}|_{(\boldsymbol{I}-\boldsymbol{T})V} = -\boldsymbol{I}$. *A subspace* $H \subset V = F_q^n$ *is called the invariant subspace of* $\boldsymbol{T}$ *if* $H = \ker(\boldsymbol{I}-\boldsymbol{T})$, *and the subspace* $L = \mathrm{Im}(\boldsymbol{I}-\boldsymbol{T}) \subset V$ *is called the line subspace of* $\boldsymbol{T}$.

Remark 1.1 (1) Given an i-transvection $\boldsymbol{T}$ with a invariant subspace H and a line subspace L, since $\boldsymbol{T}|_{(\boldsymbol{I}-\boldsymbol{T})V} = \boldsymbol{I}$ and $(\boldsymbol{I}-\boldsymbol{T})V = \mathrm{Im}(\boldsymbol{I}-\boldsymbol{T})$, it yields that $\mathrm{Im}(\boldsymbol{I}-\boldsymbol{T}) \subseteq \ker(\boldsymbol{I}-\boldsymbol{T})$, i.e., $L \subseteq H$.