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The Mod 2 Kauffman Bracket Skein Module of Thickened Torus

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Abstract: Framed links in thickened torus are studied. We define the mod 2 Kauffman bracket skein module of thickened torus and give an expression of a framed link in this module. From this expression we propose a new ambient isotopic invariant of framed links.

Key words: framed link, thickened torus, mod 2 Kauffman bracket skein module

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1 Introduction

We are concerned with framed links in thickened torus $T^2 \times I$ by using skein theory. We will extend the Kauffman bracket skein module to the mod 2 Kauffman bracket skein module and obtain an expression of a framed link as a new ambient isotopic invariant.

Skein relations have their origin in an observation by Alexander[1], Conway found a way to calculate the Alexander polynomial of a link using a so-called skein relation[2]. This is an equation that relates the polynomial of a link to the polynomial of links obtained by changing the crossings in a projection of the original link. Skein modules were introduced by Przytycki in [3]. Skein modules are quotients of free modules over ambient isotopy classes of framed links in a 3-manifold by properly chosen local skein relations. The skein module based on Kauffman bracket skein relation is one of the most extensively studied object of the algebraic topology based on framed links, which is also an important invariant of 3-manifolds. There have been extensive study and application of Kauffman bracket skein

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module (see [4]–[8]).

A convenient way of representing a framed link in an orientable 3-manifold M is in the form of smoothly embedded closed bands $(\sqcup_{j=1}^{k} S_j^1 \times I \hookrightarrow M)$, such that bands for different components do not intersect. For a framed link, let ϕ_j be the linking number between the knots $S_j^1 \times \{0\} \hookrightarrow M$ and $S_j^1 \times \{1\} \hookrightarrow M$ associated with each component of the framed link, $j = 1, \cdots, k$, we call $(\phi_1, \cdots, \phi_k)$ the framing of the framed link.

If we work with regular projections of links, then the topology of links is reflected by Reidemeister moves. Regular isotopy is the equivalent relation on link projections generated by the Reidemeister moves of types II and III. The Reidemeister moves of types II and III on the cores of bands extend to the bands themselves, while the type I move dose not extend (it corresponds to a full twist on the band). Consequently, regular isotopy corresponds to ambient isotopy of framed links.

Noted that torus knot is a kind of knot that had been investigated and used widely (see [9]). We are concerned in this paper with the torus knot, which is defined below. Given two generators x_1, x_2 in $\pi_1(T^2)$, where

$$x_1 : S^1 \hookrightarrow T^2, \quad x_1(\mathrm{e}^{\mathrm{i}\theta}) = (\mathrm{e}^{\mathrm{i}\theta}, 1),$$
$$x_2 : S^1 \hookrightarrow T^2, \quad x_2(\mathrm{e}^{\mathrm{i}\theta}) = (1, \mathrm{e}^{\mathrm{i}\theta}),$$

and consider the closed curve

$$\gamma : S^1 \hookrightarrow T^2, \quad \gamma(\mathrm{e}^{\mathrm{i}\theta}) = x_1^p x_2^q.$$

If $(p, q) = (0, 0)$ or p, q are relatively prime, then γ is called a (p, q) knot in T^2, denoted by $K_{(p,q)}$. Obviously,

$$K_{(1,0)} = x_1, \qquad K_{(0,1)} = x_2.$$

This paper is organized by two sections: In Section 2, we cover the necessary definitions and lemmas. The main result and its proof are provided in Section 3.

2 Preliminary

The data that determine a knot in $\mathbf{R}^3$ are usually given by a projection onto a plane. Now we derive it in thickened surface $F \times I$ as in $\mathbf{R}^3$.

Definition 2.1[10] *Let F be a compact orientable surface, L be a framed link in the thickened surface $F \times I$. Suppose $r : \sqcup_{j=1}^{k} S_j^1 \times \{0\} \to \sqcup_{j=1}^{k} S_j^1 \times I$, $p : F \times I \to F \times \{0\}$, we call the composition mapping $p \cdot L \cdot r : \sqcup_{j=1}^{k} S_j^1 \to F$ a projection of L onto F, denoted by ℓ.*

Definition 2.2[10] *A projection ℓ of a framed link L is called regular if*

(1) *ℓ is an immerse;*

(2) *there are only finitely many intersections in ℓ and all intersections are double points;*

(3) *ℓ is transverse to the every intersection point.*

Moreover, if the upper crossing line and the lower crossing line are marked at every double point in a regular projection, then this regular projection of a link is called a link projection.