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Energy Estimate Related to a Hardy-Trudinger-Moser Inequality

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Abstract. Let $\mathbb{B}_1$ be a unit disc of $\mathbb{R}^2$, and $\mathscr{H}$ be a completion of $C_0^\infty(\mathbb{B}_1)$ under the norm

$$\|u\|_{\mathscr{H}}^2=\int_{\mathbb{B}_1}\left(|\nabla u|^2-\frac{u^2}{(1-|x|^2)^2}\right)\mathrm{d}x.$$

Using blow-up analysis, Wang-Ye [1] proved existence of extremals for a Hardy-Trudinger-Moser inequality. In particular, the supremum

$$\sup_{u\in\mathscr{H},\|u\|_{\mathscr{H}}\le 1}\int_{\mathbb{B}_1}e^{4\pi u^2}\mathrm{d}x$$

can be attained by some function $u_0\in\mathscr{H}$ with $\|u_0\|_{\mathscr{H}}=1$. This was improved by the author and Zhu [2] to a version involving the first eigenvalue of the Hardy-Laplacian operator $-\Delta-1/(1-|x|^2)^2$. In this note, the results of [1,2] will be reproved by the method of energy estimate, which was recently developed by Malchiodi-Martinazzi [3] and Mancini-Martinazzi [4].

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1 Introduction

Let $\mathbb{B}_1\subset\mathbb{R}^2$ be a unit disc, $C_0^\infty(\mathbb{B}_1)$ include all smooth functions with compact support in $\mathbb{B}_1$, and $\mathscr{H}$ be a closure of all functions $u\in C_0^\infty(\mathbb{B}_1)$ under the norm

$$\|u\|_{\mathscr{H}}^2=\int_{\mathbb{B}_1}\left(|\nabla u|^2-\frac{u^2}{(1-|x|^2)^2}\right)\mathrm{d}x.$$

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It was proved by Wang-Ye [1] that

$$\sup_{u\in\mathscr{H},\|u\|_{\mathscr{H}}\leq 1}\int_{\mathbb{B}_1} e^{4\pi u^2}\mathrm{d}x<\infty; \tag{1.1}$$

moreover, the supremum can be attained by some function $u_0\in\mathscr{H}$ with $\|u_0\|_{\mathscr{H}}=1$. Let $\lambda_1(\mathbb{B}_1)=\inf_{u\in\mathscr{H},u\not\equiv 0}\|u\|^2_{\mathscr{H}}/\|u\|_2^2$ be the first eigenvalue of the Hardy-Laplacian operator $-\Delta-1/(1-|x|^2)^2$, where $\|\cdot\|_2$ denotes the $L^2(\mathbb{B}_1)$-norm. Later joined with Zhu [2], the author improved the Hardy-Trudinger-Moser inequality (1.1) to the following version: For any $\alpha<\lambda_1(\mathbb{B}_1)$, there holds

$$\sup_{u\in\mathscr{H},\|u\|^2_{\mathscr{H}}-\alpha\|u\|_2^2\leq 1}\int_{\mathbb{B}_1} e^{4\pi u^2}\mathrm{d}x<\infty; \tag{1.2}$$

also extremals of the above supremum exist. Here for earlier works on Trudinger-Moser inequalities involving eigenvalues, we refer the readers to [5–10]. Both methods employed in articles [1,2] are blow-up analysis, which was previously used by Carleson-Chang [11], Ding-Jost-Li-Wang [12], Adimurthi-Struwe [13] and Li [14]. According to [1,2], for any $\alpha<\lambda_1(\mathbb{B}_1)$ and any $k\in\mathbb{N}$, there exists a decreasingly radially symmetric function $u_k\in\mathscr{H}$ such that

$$\int_{\mathbb{B}_1}\exp((4\pi-1/k)u_k^2)\mathrm{d}x=\sup_{u\in\mathscr{H},\|u\|^2_{\mathscr{H}}-\alpha\|u\|_2^2\leq 1}\int_{\mathbb{B}_1} e^{(4\pi-1/k)u^2}\mathrm{d}x, \tag{1.3}$$

and that u_k solves

$$\begin{cases} -\Delta u_k-\frac{u_k}{(1-|x|^2)^2}-\alpha u_k=\lambda_k u_k\exp((4\pi-1/k)u_k^2) \\ u_k\geq 0 \quad \text{in} \quad \mathbb{B}_1 \\ \|u_k\|^2_{\mathscr{H}}-\alpha\|u_k\|_2^2=1. \end{cases} \tag{1.4}$$

For simplicity, we write $v_k=\sqrt{4\pi-1/k}\,u_k$. Then $v_k:\mathbb{B}_1\to\mathbb{R}$ is decreasingly radially symmetric and satisfies

$$\begin{cases} -\Delta v_k-\frac{v_k}{(1-|x|^2)^2}-\alpha v_k=\lambda_k v_k\exp(v_k^2) \\ v_k\geq 0 \quad \text{in} \quad \mathbb{B}_1 \\ \|v_k\|^2_{\mathscr{H}}-\alpha\|v_k\|_2^2=4\pi-1/k. \end{cases} \tag{1.5}$$

In particular, $v_k(0)=\max_{\mathbb{B}_1} v_k$ and $v_k(x)=v_k(r)$ for all $|x|=r$. Here and in the sequel we slightly abuse notation $u(r)$ for any radially symmetric function u defined on $\mathbb{B}_R$ or $\mathbb{R}^2$.

Our aim is to reprove results of [1,2] via the method of energy estimate, which was originated by Malchiodi-Martinazzi [3] and Mancini-Martinazzi [4]. Related problems are referred to [15–17]. We shall estimate the energy of v_k as follows.