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Some Estimates of the Maximum Modulus for Polynomials with Gaps

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Abstract. Let $p(z)$ be a polynomial of degree n having some zeros at a point $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and the rest of the zeros lying on or outside the boundary of a prescribed disk. In this brief note, we consider this class of polynomials and obtain some bounds for $\left(\max_{|z|=R} |p(z)|\right)^s$ in terms of $\left(\max_{|z|=1} |p(z)|\right)^s$ for any $R \geq 1$ and $s \in \mathbb{N}$.

Key Words: Polynomials, maximum modulus, zeros, prescribed disk.

AMS Subject Classifications: 30C10, 30C80, 30D15, 26C10, 26D10

1 Introduction

Let $p(z)$ be a polynomial of degree n. For effective management of space, we shall adopt the following notations:

$$D(0,k) := \{z : |z| < k\}, \qquad S(0,k) := \{z : |z| = k\}, \qquad M(p,R) := \max_{|z|=R} |p(z)|,$$
$$m(p,k) := \min_{|z|=k} |p(z)|, \qquad ||p|| := \max_{|z|=1} |p(z)|,$$

where k and R are positive real numbers.

By using the maximum modulus principle, one obtains that for $R \geq 1$,

$$M(p,R) \geq ||p||.$$

The general problem of interest, however, is the following:

$$\text{(P)}: \quad \text{Find a factor } (*) \text{ such that } M(p,R) \leq (*)\, ||p|| \text{ for any } R \geq 1.$$

In view of (P), S. Bernstein [6, pp. 442] observed that for $R \geq 1$,

$$M(p,R) \leq R^n ||p||. \tag{1.1}$$

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The above result is best possible with equality holding for $p(z) = \lambda z^n$, λ being a complex number. Since the extremal polynomial $p(z) = \lambda z^n$ in (1.1) has all its zeros at the origin. It should be possible to improve upon the bound in (1.1) for polynomial not vanishing at the origin. For this, Ankeny and Rivlin [1] proved that if $p(z)$ has no zero in $D(0,1)$, then for $R \geq 1$,

$$M(p,R) \leq \frac{R^n+1}{2}||p||. \tag{1.2}$$

As a sharpening of the above result, Aziz and Dawood [3] proved that for $R \geq 1$,

$$M(p,R) \leq \frac{R^n+1}{2}\,||p|| - \frac{R^n-1}{2}m(p,1). \tag{1.3}$$

Now, for the class of polynomials not vanishing in the disk $D(0,k)$, $k \geq 1$, Shah [10] proved that if $p(z)$ is a polynomial of degree n having no zero in $D(0,k)$, $k \geq 1$, then for every real number $R > k$,

$$M(p,R) \leq \frac{R^n+1}{1+k}\,||p|| - \frac{R^n-1}{1+k}\,m(p,k). \tag{1.4}$$

Several research articles have been written on this subject of inequalities (see for example Govil and Mohapatra [4], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5].)

Inspired by the work in [8], we consider polynomials having some zeros at a point $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and the rest of the zeros lying on or outside the boundary of a prescribed disk. For this, we estimate $(M(p,R)/||p||)^s$ for any $R \geq 1$ and any natural number s. The paper is organized as follows: we present two lemmas in Section 2 which will be used in the proof of our results. In Section 3, the results are formulated and proved and then followed by a short conclusion in Section 4.

2 Lemmas

For the proof of our theorems, we will need the following lemmas due to Nakprasit and Somsuwan [7].

Lemma 2.1. *Let*

$$p(z) = (z-z_0)^m \left(a_0 + \sum_{j=\mu}^{n-m} a_j z^j\right), \quad 1 \leq \mu \leq n-m, \quad 0 \leq m \leq n-1,$$

be a polynomial of degree n having zero of order m at z_0 with $|z_0| < 1$ and the remaining $n-m$ zeros are outside $D(0,k)$, $k \geq 1$. Then

$$\max_{|z|=1}|p'(z)| \leq \left[\frac{m}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^m}\right]||p|| - \frac{A}{(k+|z_0|)^m}m(p,k),$$