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RECOVERY BASED FINITE ELEMENT METHOD FOR BIHARMONIC EQUATION IN 2D*

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Abstract

We design and numerically validate a recovery based linear finite element method for solving the biharmonic equation. The main idea is to replace the gradient operator ∇ on linear finite element space by $G(\nabla)$ in the weak formulation of the biharmonic equation, where G is the recovery operator which recovers the piecewise constant function into the linear finite element space. By operator G, Laplace operator Δ is replaced by $\nabla \cdot G(\nabla)$. Furthermore, the boundary condition on normal derivative $\nabla u \cdot \boldsymbol{n}$ is treated by the boundary penalty method. The explicit matrix expression of the proposed method is also introduced. Numerical examples on the uniform and adaptive meshes are presented to illustrate the correctness and effectiveness of the proposed method.

Mathematics subject classification: 65N30.
Key words: Biharmonic equation, Linear finite element, Recovery, Adaptive.

1. Introduction

The biharmonic equation is a fourth order equation which arises in areas of continuum mechanics, including linear elasticity theory and the solution of Stokes flow. In this work, we consider a C^0 linear finite element method for the biharmonic equation in two-dimensional space.

$$\Delta^2 u(x,y) = f(x,y), \qquad \forall (x,y) \in \Omega, \tag{1.1}$$

with boundary conditions

$$u(x,y) = g_1(x,y), \qquad (x,y) \in \partial\Omega, \tag{1.2}$$

$$u_n(x,y) = g_2(x,y), \qquad (x,y) \in \partial\Omega. \tag{1.3}$$

Here Ω is a bounded domain in the two-dimensional space $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, $u_n = \nabla u \cdot \boldsymbol{n}$ is the normal derivative of u on $\partial\Omega$, and $\boldsymbol{n}$ is the unit normal vector pointing outward. The biharmonic operator Δ^2 is defined through

$$\Delta^2 = \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

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The basic idea of our method is applying the gradient recovery technique as a pre-processing tool to solve the high-order partial differential equations.

The mixed form is rewrite the biharmonic equaiton (1.1)-(1.3) into a coupled system of Poisson equations as

$$\begin{cases} \Delta v(x,y) = f(x,y), & (x,y) \in \Omega, \\ \Delta u(x,y) = v(x,y), & (x,y) \in \Omega, \\ u(x,y) = g_1(x,y), & (x,y) \in \partial\Omega, \\ u_n(x,y) = g_2(x,y), & (x,y) \in \partial\Omega. \end{cases} \tag{1.4}$$

One can easily see that under this formulation, there are two boundary conditions for the solutions u but no boundary condition for the new variable v. Thus, it is much more difficult to solve the biharmonic equation with the boundary conditions (1.2) and (1.3). These computations are dependent on the accurate evaluation of the missing boundary values for v, and the computational procedures are often unsatisfactory. The treatment of the boundary condition for the splitting method is a challenging problem since poor boundary approximations may reduce the accuracy of the numerical solution. An alternative technique is the so-called coupled equation approach,

$$\begin{cases} \Delta v(x,y) = f(x,y), & (x,y) \in \Omega, \\ v(x,y) = \Delta u(x,y) - c(u_n - g_2(x,y)), & (x,y) \in \partial\Omega, \end{cases}$$
$$\begin{cases} \Delta u(x,y) = v(x,y), & (x,y) \in \Omega, \\ u(x,y) = g_1(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

where c is a constant, see [16, 30]. For a given initial guess $v_0(x,y)$, an iteration solution $(u_k(x,y), v_k(x,y))$ can be computed until its convergence.

There are various finite element methods to discretize the biharmonic equation in the literature. As the most classical approach, the C^1 conforming finite element methods require the basis functions and their derivatives are continuous on $\bar{\Omega}$, which are rarely used in practice for their too many degrees of freedom and implementation complexity. For example, the Argyris finite element method [13] has 21 degrees of freedom for triangles. The nonconforming finite element methods such as the Adini element or Morley element [3, 13, 25, 31, 34] are popular methods for numerical solution of the high-order partial differential equations. The key idea in nonconforming methods is to use the penalty term to ensure the convergence into the natural energy space of the variational problem. Mixed finite element method is another choice which is based on the equivalent form (1.4) and only require the Lagrangian finite element spaces, which are widely used in practice, but they require very careful treatment on the essential and natural boundary conditions. The literature on the mixed finite element methods is vast, and we refer to [1, 9, 14, 22, 33, 37] and the references therein for the detail of these methods. In the case of Ω is nonconvex, the solution of the mixed numerical formulation may be spurious solution. Indeed the solution obtained from the mixed fomulation (1.4) in general does not belong to H^2, thus the corresponding mixed finite element method for (1.1)-(1.3) is problematic when Ω is nonconvex [6, 23]. The discontinuous Galerkin method is also a choice which is based on standard continuous Lagrangian finite element spaces [8, 17] or completely discontinuous finite element spaces [19, 32]. Other methods which have been developed for fourth order problems include finite difference methods [2, 11, 20], and finite volume method [18].

An alternative to the aforementioned methods is the recovery based finite element method developed in recent years [10, 12, 24, 28, 29]. It is a nonconforming finite element method based