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Explicit H^1-Estimate for the Solution of the Lamé System with Mixed Boundary Conditions

AIT-AKLI Djamel* and MERAKEB Abdelkader

L2CSP, Mouloud Mammeri University Tizi-Ouzou, 15000, Algeria.

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Abstract. In this paper we consider the Lamé system on a polygonal convex domain with mixed boundary conditions of Dirichlet-Neumann type. An explicit L^2 norm estimate for the gradient of the solution of this problem is established. This leads to an explicit bound of the H^1 norm of this solution. Note that the obtained upper-bound is not optimal.

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1 Introduction

Let Ω be a bounded open connected subset of $\mathbb{R}^2$. The static equilibrium of a deformable structure occupying Ω is governed by the Lamé linear elasto-static system, see [1]. In this paper, we restrict the study to a convex domain Ω whose boundary has a polygonal shape that posses $m+1$ edges with $m\geq 2$. We denote $\Gamma=\cup_{i=0}^{m}\Gamma_i$ its boundary and $d(\Omega)$ its diameter. Moreover, we assume that all the edges Γ_i have strictly positive measure. The system under consideration is given by

$$\begin{cases} Lu=f & \text{a.e in } \Omega, \\ \sigma(u)\cdot\overrightarrow{n_i}=g_i & \text{on } (\Gamma-\Gamma_0)\cap\Gamma_i,\ \ 1\leq i\leq m, \\ u=0 & \text{on } \Gamma_0. \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `djamel.aitakli@ummto.dz` (D. Ait-akli), `kader.merakeb@ummto.dz` (A. Merakeb)

We need to assume that the edges Γ_i which form the boundary Γ fulfill a condition similar to assumption (H_2) in ([2], Theorem 2.3). Actually, for our purpose, a stronger condition is needed and it is formulated in (1.5) below. The vector function $u=(u^1,u^2)$ satisfying the system (1.8) describes a displacement in the plane. In this model we impose a homogeneous Dirichlet condition on Γ_0 and a Neumann condition on the remaining part of the boundary. The equality on the boundary is understood in the sense of the trace. We denote L the Lamé operator defined by

$$Lu := -\operatorname{div}\sigma(u) = -\operatorname{div}[2\mu\varepsilon(u)+\lambda \operatorname{Tr}\varepsilon(u) Id]. \tag{1.2}$$

We assume the data functions f and g at the right hand sides to satisfy $f\in[L^2(\Omega)]^2$ and $g\in[H^{\frac{1}{2}}(\Gamma-\Gamma_0)]^2$. The vector $\overrightarrow{n_i}$ represents the outside normal to Γ_i. We write μ and λ the Lamé's coefficients. We place ourselves in the isotropic framework, the deformation tensor ε is defined by

$$\varepsilon(u)=\frac{1}{2}(\nabla u+\nabla^t u). \tag{1.3}$$

The weak form of problem (1.1) is (see [1,3]): Find $u\in V$ such that $\forall v\in V$

$$\int_\Omega 2\mu\varepsilon(u)\varepsilon(v)+\lambda \operatorname{div}u \ \operatorname{div}v \mathrm{d}x=\int_\Omega f v\mathrm{d}x+\int_{\Gamma-\Gamma_0} g v \mathrm{d}\sigma(x), \tag{1.4}$$

where

$$V=\Big\{v\in[H^1(\Omega)]^2;\quad v=0 \quad \text{on } \Gamma_0\Big\}.$$

The existence and uniqueness issue of the solution of (1.4) in V is classic, (see [3]).

If we denote θ the interior angle between the edges Γ_j and Γ_k, $0\le j,k\le m$ such that $\overline{\Gamma}_j\cap\overline{\Gamma}_k\neq\emptyset$ and if we denote γ the interior angle between the Neumann part of the boundary $\Gamma_N:=\Gamma-\Gamma_0$ and the Dirichlet part of the boundary $\Gamma_D:=\Gamma_0$, then we impose

$$0<\theta<\pi, \qquad 0<\gamma<\pi. \tag{1.5}$$

The reason behind this assumption on the boundary is to get a better regularity of the solution of the weak problem (1.4). Precisely in that case we have, following ([2], Theorem 2.3) stated at the bottom of page 330, $u\in[H^{\frac{3}{2}+\iota}(\Omega)]^2$ for some positive $\iota>0$, which implies in particular, using the appropriate Sobolev embedding and since Ω is a locally Lipschitz domain, see part II of ([4], Theorem 4.12, page 85), that $u\in[C^{0,\frac{1}{2}+\iota}(\overline{\Omega})]^2$ i.e. u is $(\frac{1}{2}+\iota)-$holder continuous. One should notice that condition (1.5) are met since the domain considered in our case is convex. Let us denote

$$||\varepsilon(u)||_{0,\Omega}:=\left(\int_\Omega \varepsilon(u)\varepsilon(u)\mathrm{d}x\right)^{\frac{1}{2}}; \qquad ||\nabla u||_{0,\Omega}:=\left(\int_\Omega |\nabla u^1|^2+|\nabla u^2|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

By using the second Korn inequality, see [5], the trace and the Poincaré's inequalities, one easily gets from (1.4) the following estimate

$$||\nabla u||_{0,\Omega}\le\frac{1}{c_k}\frac{1}{2\mu}\Big(c_p||f||_{0,\Omega}+c_{p,t}||g||_{\frac{1}{2},\Gamma-\Gamma_0}\Big), \tag{1.6}$$