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ADI-Spectral Collocation Methods for Two-Dimensional Parabolic Equations

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Abstract. ADI-spectral collocation methods for two-dimensional parabolic equations on bounded and unbounded domains are studied. A spectral collocation scheme is adopted for spatial discretisation and the Crank-Nicolson ADI scheme is used for time discretisation. Numerical results show the stability and efficiency of the proposed collocation schemes in solving high-dimensional time-dependent problems.

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Key words: ADI-spectral collocation methods, two-dimensional parabolic equations, bounded and unbounded domains.

1. Introduction

Spectral methods play important role in approximate solution of differential and integral equations — cf. [1, 4, 5, 7, 8, 10, 11, 15] and references therein. In particular, spectral collocation methods are implemented in physical spaces and have notable advantages over spectral-Galerkin methods in the case of equations with variable coefficients and nonlinear problems. However, the resulting linear systems are plagued by ill-conditioning and suitable preconditioners are needed. Significant attempts are made in the usage of non-polynomial basis functions coming from generalised Birkhoff interpolation problems by the technique of inverting operators on bounded and unbounded domains [17–19]. Such bases can produce well-conditioned collocation schemes and offer optimal preconditioners for usual collocation schemes with an identity matrix arising in approximation of the operator $\mathscr{L}_\lambda[u] = \partial_x^2 u - \lambda^2 u$.

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The alternating direction implicit (ADI) iterative method proposed by Peaceman and Rachford [13] reduces a high-dimensional problem into a series of one-dimensional problems, thus cutting down computation cost and improving computational efficiency [3, 16]. Combined with finite differences, finite elements and spline collocation discretisation, ADI method became one of the most effective tools for solving high-dimensional time-dependent problems — cf. [2, 6, 12, 14].

The aim of this work is to introduce new ADI-spectral collocation methods for two-dimensional parabolic equations. In particular, for spatial discretisation we use the spectral collocation method with basis functions obtained by solving the generalised Birkhoff interpolation problems and for time discretisation the Crank-Nicolson ADI scheme is employed. As the result, the collocation matrix for $\mathscr{L}_\lambda[u]$ is a unit matrix for both linear and non-linear problems. Moreover, in comparison with the Crank-Nicolson scheme, this ADI-spectral collocation method is more stable and more efficient.

The rest of the paper is organised as follows. In Section 2 we recall non-polynomial basis functions based on Jacobi polynomials on bounded domains and introduce a new ADI-spectral collocation schemes for two-dimensional parabolic equations on bounded domains. Some numerical results are given to show its efficiency. In Section 3 we consider non-polynomial basis functions based on Laguerre functions on unbounded domains and ADI-spectral collocation schemes for two-dimensional parabolic equations on unbounded domains. Numerical results demonstrate a high accuracy and efficiency of the methods proposed.

2. ADI-Collocation Methods for Bounded Domains

We start with auxiliary results. Let $I=(-1,1)$, $\alpha,\beta>-1$. We consider the set

$$\mathscr{P}_N(I)=\operatorname{span}\left\{J_0^{(\alpha,\beta)}(x),J_1^{(\alpha,\beta)}(x),\ldots,J_N^{(\alpha,\beta)}(x)\right\}$$

of the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ of degree n — cf. [15]. Recall that they satisfy the relation

$$\int_I J_n^{(\alpha,\beta)}(x)J_{n'}^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)dx=\gamma_n^{(\alpha,\beta)}\delta_{nn'},$$

where $\delta_{nn'}$ is the Kronecker delta and

$$\begin{aligned}\omega^{(\alpha,\beta)}(x)&=(1-x)^\alpha(1+x)^\beta,\\ \gamma_n^{(\alpha,\beta)}&=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}.\end{aligned}$$

We denote by $\{x_j\}_{j=0}^N$, $x_0=-1$, $x_N=1$ the Jacobi-Gauss-Lobatto (JGL) points arranged in ascending order.

Let $\{h_j(x)\}_{j=0}^N$ be the Lagrange interpolating polynomials at the JGL points $\{x_j\}_{j=0}^N$ such that $h_j(x)\in\mathscr{P}_N(I)$ and $h_j(x_i)=\delta_{ij}$, and let $\{l_j(x)\}_{j=1}^{N-1}$ be the Lagrange interpolating