JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **33**, No. 2, pp. 109-118

doi: 10.4208/jpde.v33.n2.2
June 2020

Blow-Up Solution of the 3D Viscous Incompressible MHD System

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Received 3 February 2020; Accepted 22 February 2020

Abstract. The 3D viscous incompressible magneto-hydrodynamic (MHD) system comprised by the 3D incompressible viscous Navier-Stokes equation couples with Maxwell equation. The global well-posedness of the coupled system is still an open problem. In this paper, we study the Cauchy problem of this coupled system and establish some logarithmical type of blow-up criterion for smooth solution in Lorentz spaces.

AMS Subject Classifications: 35B40, 35B41, 35Q30, 76D03, 76D05

Chinese Library Classifications: O175.29

Key Words: MHD equations; blow-up criteria; Lorentz space.

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1 Introduction

The 3D viscous incompressible magneto-hydrodynamic (MHD) system describes the magnetic properties of electrically conducting fluids. It is a strongly coupled PDE system that combines the Navier-Stokes equation of fluid dynamics together with Maxwell's equations of electro-magnetism. Many problems related to the individual component of the coupled system remain open. For example, the well-posedness of in 3D for the Navier-Stokes equation, that describes the motion of classical fluid flow, is listed as one of the Millennium problem by the Clay Mathematics Institute. Therefore, the global well-posedness for the MHD system in 3D is also open problem. In this paper, we want to study blow-up criterion for smooth solution to the Cauchy problem of the MHD system

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in 3D where the underlying fluid is viscous and incompressible. The coupled system in the variables of u and b is given as the following

$$\begin{cases} u_t+u\cdot\nabla u-\nu\Delta u+\nabla p=b\cdot\nabla b, & x\in\mathbb{R}^3,\ t>0,\\ b_t+u\cdot\nabla b-\eta\Delta b=b\cdot\nabla u, & x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u=\nabla\cdot b=0, & x\in\mathbb{R}^3,\ t>0,\\ u(x,0)=u_0(x),\ \ b(x,0)=b_0(x), & x\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u=(u_1,u_2,u_3)\in\mathbb{R}^3$ denotes the velocity field, $b=(b_1,b_2,b_3)\in\mathbb{R}^3$ is the magnetic field, $p\in\mathbb{R}$ is the scalar pressure, $\nu>0$ is the viscosity, $\eta>0$ is the magnetic diffusivity, u_0 and b_0 are the initial velocity field and magnetic field satisfying $\nabla\cdot u_0=0$ and $\nabla\cdot b_0=0$ in the distributional sense.

For system (1.1), there are many results available in the literature pertaining to the well-posedness and regularity of the global solution. For instance, the famous Beale-Kato-Majda criterion [1] gives conditions to the well-posedness of a global smooth solution of (1.1) when the magnet field b is a constant and $\nu=0$. In this case, (1.1) reduces to the classical incompressible Euler equation and the BKM criterion describes the control of vorticity for the fluid $\omega=\text{curl}u$ in $L^1(0,T;L^\infty)$. For more details, one can refer to Majda and Bertozzi [2]. If the magnet field b is a constant, then (1.1) reduces to the classical incompressible Navier-Stokes equation, of which many blow-up results for smooth solutions in terms of pressure or the velocity field are available. One could refer to Serrin [3], Zhou [4–6], Fan, et al. [7], Chae and Lee [8], Zhou and Lei [9], Cao and Titi [10], Kato and Ponce [11], Breselli and Galdi [12], Lemarié-Rieusse [13] and references therein.

Since the global well-posedness of the 3D Navier-Stokes equation is unknown, the solution of MHD flow should not have more regularity than that of Navier-Stokes equation. The global well-posedness of the 3D MHD equations, including the regularity criteria for the strong and weak solutions, have also drawn a lot of attentions. We review some of the blow-up criterion of smooth solutions in terms of the velocity field or pressure: in [14], Cao and Wu gave two regularity criteria of strong solutions in Sobolev spaces; He and Xin in [15] derived a regularity criterion of the velocity field for the weak solutions; Gala [16] investigated the blow-up criteria in multiplier spaces. For more interesting results, we may refer to Chen, et al. [17], Fan, et al. [7], Jia and Zhou [18], Zhou [19] and literature therein.

As far as we know, there are no results in the literature that are concerned about blow-up criterions for smooth solution to (1.1) in *Lorentz space* of logarithmical type. This is our main motivation for this article. The Lorentz spaces $L^{p,q}$ (see detailed definition in Section 2) are generalization of the L^p-spaces and the norm of a function in a Lorentz space provides control in the spread of a measurable function in both the horizontal and vertical directions by rescaling the measure in both the range and the domain of the function. Thus, our results are generalization of similar results in the L^p space for the Navier-Stokes equation and MHD equation. We compare these results in more details in