Journal of Computational Mathematics
Vol.38, No.4, 2020, 661–682.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1903-m2019-0125

ON NEW STRATEGIES TO CONTROL THE ACCURACY OF WENO ALGORITHM CLOSE TO DISCONTINUITIES II: CELL AVERAGES AND MULTIRESOLUTION*

Sergio Amat and Juan Ruiz[1)]
Department of Applied Mathematics and Statistics, Universidad Politécnica de Cartagena, Spain
Email: sergio.amat@upct.es, juan.ruiz@upct.es
Chi-Wang Shu
Division of Applied Mathematics, Brown University, USA
Email: Chi-Wang_Shu@brown.edu

Abstract

This paper is the second part of the article and is devoted to the construction and analysis of new non-linear optimal weights for WENO interpolation capable of rising the order of accuracy close to discontinuities for data discretized in the cell averages. Thus, now we are interested in analyze the capabilities of the new algorithm when working with functions belonging to the subspace $L^1 \cap L^2$ and that, consequently, are piecewise smooth and can present jump discontinuities. The new non-linear optimal weights are redesigned in a way that leads to optimal theoretical accuracy close to the discontinuities and at smooth zones. We will present the new algorithm for the approximation case and we will analyze its accuracy. Then we will explain how to use the new algorithm in multiresolution applications for univariate and bivariate functions. The numerical results confirm the theoretical proofs presented.

Mathematics subject classification: 65D05, 65D17, 65M06, 65N06.
Key words: WENO, Cell averages, New optimal weights, Multiresolution schemes, Improved adaption to discontinuities, Signal processing.

1. Introduction

This work is the second part of the article [1]. The first part was dedicated to the design and analysis of a new WENO algorithm for the approximation of piecewise smooth functions discretized in the point value setting. In [2] we generalized the construction introduced in [1] to any order $2r$. Our first objective in this article is to consider piecewise smooth functions with jumps belonging to the subspace $L^1 \cap L^2$. For these kind of functions, a discretization in the point values might not make sense, while a discretization through the cell averages always does. This is due to the fact that a function f with an isolated discontinuity can not be recovered from the samples $f(kh)$, being h the grid-spacing, as the exact position of the discontinuity is lost during the discretization process. Instead, the sampling should be replaced by a local average as we shall see later on. The cell average discretization arises naturally in applications where we want to recover a piecewise smooth function from its samples. Our second objective is to use the resulting approximation scheme to the design of a nonlinear and adaptive multiresolution

* Received May 20, 2019 / Revised version received February 19, 2020 / Accepted March 2, 2020 / Published online May 19, 2020 /
1) Corresponding author

scheme and to check its compression capabilities with univariate and bivariate piecewise smooth functions with jump discontinuities.

In [3] the authors introduced the WENO (Weighted ENO) algorithm with the aim of obtaining a higher order of accuracy at smooth zones using the same information as ENO (Essentially Non Oscillatory) algorithm [4,5]. WENO algorithm is basically a nonlinear convex combination of the interpolants constructed using the different stencils that ENO algorithm considers. The weights of this combination can be designed making use of smoothness indicators, that are built using divided differences. Inspired by the measure of the total variation, the authors introduced in [6] a more suitable computation of the smoothness of a function. The nonlinear weights constructed using these smoothness indicators assure that the stencils that cross a discontinuity have a very small contribution to the final approximation. An incomplete list of references about WENO is [7–10]. The references [11,12] are particularly interesting as they present a review of the state of the art about WENO algorithms.

In [1] we presented a redesign of the WENO weights that allows to increase the accuracy of the WENO algorithm close to discontinuities in the point values. A generalization of the algorithm was presented in [2].The main improvement presented in [1,2] is the design of data dependent nonlinear optimal weights, which are considered constant in the classical WENO algorithm. Through these weights we proved in [1] that we can improve the results of classical WENO algorithm so that the accuracy close to the discontinuities is reduced step by step, optimizing the accuracy while, at the same time, maintaining the essentially non oscillatory property.

The characteristics of the reconstruction process in Harten's general framework for multiresolution makes it relatively simple to design an adaptive multiresolution scheme. It is known that, in order to obtain an adaptive multiscale decomposition, we just have to include the adaptivity in the reconstruction process. Through this adaptivity and using approximation theory, it is possible to predict that nonlinear and adaptive reconstruction schemes, that are capable of taking into account the local characteristics of the data, are susceptible of being used to design nonlinear and adaptive multiresolution schemes with better compression properties. WENO algorithm is a very good example of nonlinear reconstruction scheme with data dependency. These considerations lead us to study the capabilities of the scheme presented in [1,2] for multiresolution applications for piecewise smooth data discretized in the cell averages.

This paper is organized as follows: In Section 2 we introduce Harten's multiresolution for data discretized in the cell averages. Section 3 exposes how the classical WENO algorithm for the cell averages is constructed. Section 4 is dedicated to explain how to construct the new algorithm in the cell averages and to theoretically analyze its accuracy. Section 5 presents some experiments dedicated to numerically analyze the accuracy of the new algorithm through a grid refinement analysis close to discontinuities. We also present examples of application of the new algorithm to the compression of univariate and bivariate functions. Finally, Section 6 presents the conclusions.

2. Harten's Multiresolution for the Cell Averages

In this section we introduce Harten's framework for multiresolution in the cell averages, as some of the applications presented in the last part of the article will make use of it. A nice discussion about Harten's multiresolution is presented in [13].