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AN ADAPTIVE VISCOSITY E–SCHEME FOR DEGENERATE CONSERVATION AND BALANCE LAWS

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Abstract. An adaptive E–scheme for degenerate, viscous balance laws is presented. Taking into account natural diffusion, numerical viscosity is locally reduced to a minimum. Numerical experiments demonstrate the improved accuracy of the adaptive scheme. Explicit and implicit three–point E–schemes are monotone, TVD and nonlinearly stable. A high–resolution version of the adaptive E–scheme is derived and tested in experiments. The latter is not necessarily monotone, but TVD.

Key words. Balance law, E–scheme, monotone, degenerate viscosity.

1. Introduction

For scalar conservation laws $u_t + f(u)_x = 0$ the entropy solution may be constructed as the vanishing viscosity weak solution to the viscous conservation law $u_t + f(u)_x = du_{xx}$. Numerical methods for hyperbolic conservation laws appeal to that principle by using artificial, vanishing viscosity. For example, the classical Lax–Friedrichs scheme applies the numerical viscosity $d = \epsilon\Delta x$ where $\epsilon = 0.5\Delta x/\Delta t$. By the CFL–condition ϵ is bounded from below $2\epsilon \geq \|f'\|_\infty$. In agreement with the concept: sufficient diffusion grants stability, Tadmor [25] showed that any scheme containing more numerical viscosity than Godunov's scheme is entropy–stable. Moreover, it is exactly the class of E–schemes [21] that have no less numerical viscosity than that of Godunov.

In this paper we present an adaptive viscosity E–scheme for degenerate, viscous conservation laws

$$u_t + f(u)_x = (d(u,x)u_x)_x \quad , \quad d(u,x) \geq 0 \tag{1}$$

making use of the given "natural" diffusion $d(u,x) \geq 0$ and adding only that much numerical viscosity as needed for stability. The resulting adaptive viscosity scheme is an E–scheme. In Sect. 5 we prove that explicit three-point E–schemes are monotone. Applying the calculus of inverse–monotone matrices, it is shown in Sect. 6 that also implicit E–schemes are monotone. Using Kröner's version [18] of Harten's theorem [10], it follows that ϑ–time stepping with E–fluxes is a TVD operation, see Sect. 7. Numerical experiments in Sect. 9 demonstrate the effect of reduced numerical viscosity in the presence of natural diffusion. Finally, in Sect. 13 a nonlinear reaction term is included in the analysis and stability of the adaptive E–scheme when applied to balance laws is proven. Conclusions follow in Sect. 14.

The numerical analysis of possibly degenerate convection–diffusion equations has a long history. Some milestones are the following: Crandall and Majda [5] studied monotone schemes. Breuß[2, 3] presented a rigorous theory of implicit, monotone methods. Osher [21] introduced E–schemes and Tadmor [25] showed their entropy stability by comparison to the classical Godunov scheme. Even so these schemes are designed for hyperbolic conservation laws, their analysis relies on numerical viscosity. Karlsen et al. in a series of papers [7, 6, 4, 12, 13, 14] developed the theory and numerics of strongly degenerate convection–diffusion equations. In [7] they found that strongly degenerate

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problems develop more complex structures than purely hyperbolic equations. A class of conservative–form difference schemes, treating the convective and diffusive flux as one effective conservative flux, is shown to converge to the unique BV entropy weak solution. Our contribution is to minimize the numerical diffusion in the effective flux (without solving nonlinear Riemann problems). In [4] Chen and Karlsen establish continuous dependence for anisotropic degenerate parabolic PDEs. Anisotropic numerical viscosity appears in finite volume schemes on unstructured grids where total variation bounds are not available. Error estimates and convergence rates are given in [12, 13, 14].

2. Preliminaries

Consider the convection–diffusion equation (1) with $f \in C^1(\mathbb{R})$, $\|f'\|_\infty < \infty$ and possibly degenerate "natural" diffusion $d(u,x) \geq 0$. Let $\epsilon\Delta x \geq 0$ denote artificial diffusion and $D = d + \epsilon\Delta x$ the effective, total diffusion. On an uniform mesh $x_j = j\Delta x$, $\Delta x > 0$ the second order central difference operator

$$-\frac{1}{\Delta x^2}\Gamma_D \approx \partial_x\left(D\partial_x\right)$$

is given by a symmetric matrix. At inner grid points it has the local structure

$$\Gamma_D \sim \begin{pmatrix} D_{j-3/2}+D_{j-1/2} & -D_{j-1/2} & \\ -D_{j-1/2} & D_{j-1/2}+D_{j+1/2} & -D_{j+1/2} \\ & -D_{j+1/2} & D_{j+1/2}+D_{j+3/2} \end{pmatrix}$$

where $D_{j+1/2} = D(x_{j+1/2})$ is evaluated at the interface $x_{j+1/2} = (j+1/2)\Delta x$. The convection term is discretized as

$$f(u)_x \approx \frac{1}{2\Delta x}\Lambda\phi(u) \ ,$$

where ϕ denotes the diagonal field $\phi(u)_j = f(u_j)$ and Λ is the anti–symmetric, central difference operator

$$\Lambda \sim \begin{pmatrix} 0 & 1 & \\ -1 & 0 & 1 \\ & -1 & 0 \end{pmatrix} \ .$$

For periodic problems it will be sufficient to consider "inner" mesh points. Applying periodic boundary conditions and identifying overlapping points the equations at the boundary are the same, see Sect. 8. Also note the finite mesh and finite dimensional discrete operators represented by finite matrices.

The convection diffusion operator $-f(u)_x + (d(u,x)u_x)_x$ is discretized by central differences

$$F_{\Delta x}(u) = -\frac{1}{\Delta x^2}\Gamma_D u - \frac{1}{2\Delta x}\Lambda\phi(u) \ .$$

The classical, central difference scheme has no artificial diffusion $\epsilon = 0$, while Lax–Friedrichs uses $\epsilon = \frac{1}{2}\frac{\Delta x}{\Delta t}$. The forward marching scheme

$$u^{n+1} = u^n + \Delta t F_{\Delta x}(u^n) = \mathcal{H}(u^n)$$

is monotone in the sense of Crandall and Majda [5] if $\mathcal{H} = I + \Delta t F_{\Delta x}$ is a non–decreasing function in all unknowns. In particular, the Jacobian $DF_{\Delta x}$ is off–diagonal non–negative, or quasi–positive.

Whenever diffusion $D = d + \epsilon\Delta x$ does not depend on u, the Jacobian reads

$$DF_{\Delta x}(u) = -\frac{1}{\Delta x^2}\Gamma_D - \frac{1}{2\Delta x}\Lambda\mathrm{diag}(f'(u))$$