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Finite Element Approximation of Space Fractional Optimal Control Problem with Integral State Constraint

Zhaojie Zhou[1], Jiabin Song[1] and Yanping Chen[2,*]

[1] *School of Mathematics and Statistics, Shandong Normal University, China*
[2] *School of Mathematical Sciences, South China Normal University, China*

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Abstract. In this paper finite element approximation of space fractional optimal control problem with integral state constraint is investigated. First order optimal condition and regularity of the control problem are discussed. A priori error estimates for control, state, adjoint state and lagrange multiplier are derived. The nonlocal property of the fractional derivative results in a dense coefficient matrix of the discrete state and adjoint state equation. To reduce the computational cost a fast projection gradient algorithm is developed based on the Toeplitz structure of the coefficient matrix. Numerical experiments are carried out to illustrate the theoretical findings.

AMS subject classifications: 65N30

Key words: Finite element method, optimal control problem, state constraint, space fractional equation, a priori error estimate, fast algorithm.

1. Introduction

Fractional PDEs are widely used to model many physical process, for example, anomalous diffusion phenomena, for which the integer order differential equations fail to provide an accurate description. Two main features of fractional differential equations which impact their numerical approximation are the nonlocality of fractional differential operator, and the low regularity of the solution. The former leads to a dense coefficient matrix in the discrete scheme such as finite element method and finite difference method, and the latter results in slow convergence of the numerical solution to exact solution. Over the past decades lots of literatures are devoted to develop efficient numerical methods and algorithms for fractional PDEs, for example, finite element methods, finite volume methods and fast algorithms, see [1, 10, 13, 14, 21, 29].

*Corresponding author. *Email address:* `yanpingchen@scnu.edu.cn` (Y. Chen)

In recent years optimal control problem governed by fractional PDEs has attracted lots of attention with the rapid development of fractional calculus. The research on numerical methods or algorithms for fractional optimal control problem forms a hot topic. We refer to [2, 17, 22, 33–35] for finite element methods, [23, 31] for spectral methods, and [12] for fast algorithm. Compared with numerical methods ([16, 19, 25, 30]) for optimal control problem governed by integer order PDEs the research on fractional optimal control problem is immature. To the best of our knowledge, there are few literatures about numerical method of state constrained optimal control problem governed by fractional differential equation. In [32] spectral Galerkin approximation of time fractional optimal control problem with integral state constraint was discussed. In [4] an optimal control problem governed by fractional elliptic equation with pointwise state constrained was investigated, where the well-posedness of the control problem and the optimality condition was proved.

The aim of the present paper is to study finite element discretization of space fractional diffusion optimal control problem with integral state constraint. Let $\Omega = (0,1)$, $\Gamma = \partial\Omega$. We consider the following control problem

$$\min_{(y,u)\in\mathcal{K}\times U_{ad}} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to

$$\begin{cases} -D\left(r\,{}_0D_x^{-(2-\alpha)} + (1-r)\,{}_xD_1^{-(2-\alpha)}\right)Dy = f+u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma \end{cases} \tag{1.2}$$

and

$$\mathcal{K} = \left\{ v \in L^1(\Omega) \,\middle|\, \int_\Omega v dx \le \delta \right\}.$$

Here $y_d \in L^2(\Omega)$ is the desired state, $\gamma > 0$ is the regularization parameter, $f \in L^2(\Omega)$ is a given function, δ is a fixed constant and $U_{ad} = L^2(\Omega)$. The parameters $r \in [0,1]$ and $\alpha \in (1,2)$. Since the cost functional is quadratic and the state constraint forms a convex set, the existence of a unique solution (y,u) to above optimal control problem can be guaranteed by standard control theory.

The fractional diffusion operator in (1.2) arises in a random walk process in which the jumps have an unbounded variance, i.e., Lévy process, [5]. Physically, it can also be interpreted as a nonlocal Fickian law [9]. The variational framework and finite element approximation were firstly established in [13]. The regularity of the solution to (1.2) were discussed in [14] based on the expression for the kernel of the fractional diffusion operator. The regularity of the solution to (1.2) in weighted Sobolev space and the corresponding spectral Galerkin approximation are discuss in [18].

To develop finite element approximation of (1.1)-(1.2) we firstly derive the first order optimality condition and discuss the regularity of the solution to the optimal control problem. A priori error estimates for control, state, adjoint state and lagrange multiplier are derived. Since the coefficient matrix of the discrete state and adjoint