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An Alternative Finite Difference Stability Analysis for a Multiterm Time-Fractional Initial-Boundary Value Problem

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Abstract. A fractional initial-boundary value problem is considered, where the differential operator includes a sum of Caputo temporal derivatives, and the solution has a weak singularity at the initial time $t = 0$. The problem is solved numerically by a finite difference method based on applying the L1 method to discretise each temporal derivative on a graded mesh. Stability of this method is proved by generalising the analysis of Stynes *et al.*, SIAM J. Numer. Anal. 55 (2017), where the case of a single temporal derivative was investigated. This stability result is used to prove a sharp error estimate for the finite difference method.

AMS subject classifications: 65M12

Key words: Multiterm time-fractional initial-boundary value problem, graded mesh, L1 scheme, discrete stability.

1. Introduction

The numerical solution of time-fractional initial-boundary value problems has been considered in a large number of recent papers (see [2] for a detailed survey). Typical solutions of such problems exhibit a weak singularity at the initial time $t = 0$, as discussed in [7–9]. Until now, few papers have considered problems with solution singularities where the differential operator contains a sum of time-fractional derivatives, although such formulations offer greater flexibility in modelling. It is this class of problems that is the subject of our paper.

The problem that we study is the following. Set $\Omega = (0, l) \subset \mathbb{R}$, with $\bar{\Omega} = [0, l]$ and boundary $\partial\Omega = \{0, l\}$. Let $T > 0$ be fixed. Set $Q = (0, l) \times (0, T]$ and $\bar{Q} = [0, l] \times [0, T]$.

For constant $\alpha \in (0, 1)$ and any suitable function $w(x, t)$ defined on $\bar{Q}$, define the temporal Caputo fractional derivative of order α of w by

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$$\partial_t^{\alpha} w(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial w(x,s)}{\partial s}\, ds.$$

Let ℓ be a positive integer. Assume that we are given constants α_i (for $i = 1, 2, \ldots, \ell$) which satisfy $0 < \alpha_\ell < \ldots < \alpha_2 < \alpha_1 < 1$. These α_i are the orders of our fractional derivatives.

We shall consider the multiterm time-fractional initial-boundary value problem

$$\sum_{i=1}^{\ell} \left[q_i \partial_t^{\alpha_i} u(x,t) \right] + Lu(x,t) = F(x,t) \quad \text{for} \quad (x,t) \in Q \tag{1.1a}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \qquad \text{for} \quad x \in \Omega, \tag{1.1b}$$

$$u(0,t) = u(l,t) = 0 \quad \text{for} \quad 0 < t \le T, \tag{1.1c}$$

where the given constants q_i are positive, $F \in C(\bar{Q})$, and $u_0 \in C(\bar{\Omega})$ with $u_0(0) = u_0(l) = 0$. Without loss of generality we assume that $q_1 = 1$. In (1.1a) the operator L is $Lu(x,t) = -u_{xx}(x,t) + c(x)u(x)$ for $(x,t) \in \Omega$, where $c \in C(\bar{\Omega})$ with $c \ge 0$.

The single-term case $\ell = 1$ of (1.1) was investigated in [9], and the main part of our paper is a generalisation of results from there.

After further regularity and compatibility conditions are imposed on the data of the problem, existence of a solution to (1.1) for the case $F \equiv 0$ follows from [4, Theorem 2.1], and for the case $u_0 \equiv 0$ from [4, Theorem 2.2]. Combining these results yields existence of a solution for (1.1). Uniqueness of this solution follows from [6, Theorem 4].

In [1] it is shown that, provided that the data of the problem satisfy certain regularity and compatibility conditions, the solution u of (1.1) satisfies

$$\left| \frac{\partial^k u}{\partial x^k}(x,t) \right| \le C \qquad \text{for} \quad k = 0,1,2,3,4, \tag{1.2a}$$

$$\left| \frac{\partial^m u}{\partial t^m}(x,t) \right| \le C\left(1 + t^{\alpha_1 - m}\right) \quad \text{for} \quad m = 0,1,2, \tag{1.2b}$$

for all $(x,t) \in (0,l) \times (0,T]$, where C is some fixed constant. The bound (1.2b) indicates the nature of the singularity in the solution u at $t = 0$.

Remark 1.1. To discern the structure of u, one can imitate the analysis of [5]. In the notation of that paper one has $n = 1$; taking $\theta = 0$ in [5, equation (5)] leads to a representation of $u - u_0$ as the solution of a weakly singular Volterra integral equation given in [5, equations (12) and (26)], where the kernel of the integral operator is a finite sum of terms (one for each derivative $\partial_t^{\alpha_i} u$). Then [5, Theorem 3] reveals the complicated structure of $u(x,t) - u_0(x)$, which we now describe. Set $\Delta := \{0, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3, \ldots, \alpha_1 - \alpha_\ell\}$. Then for some positive integer m whose value depends on the smoothness and compatibility of the data of (1.1), for each x one has

$$u(x,t) = u_0(x) + \sum_{(j,k)_\delta} \gamma_{j,k} t^{\alpha_1 + \delta_{i_0} + \delta_{i_1} + \cdots + \delta_{i_j} + k} + Z_m(t), \tag{1.3}$$