Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 2 (2020), pp. 111-127

DOI: 10.4208/ata.OA-0018

# On Well-Posedness of 2D Dissipative Quasi-Geostrophic Equation in Critical Mixed Norm Lebesgue Spaces

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Received 12 November 2019; Accepted (in revised version) 28 April 2020

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**Abstract.** We establish local and global well-posedness of the 2D dissipative quasi-geostrophic equation in critical mixed norm Lebesgue spaces. The result demonstrates the persistence of the anisotropic behavior of the initial data under the evolution of the 2D dissipative quasi-geostrophic equation. The phenomenon is a priori nontrivial due to the nonlocal structure of the equation. Our approach is based on Kato's method using Picard's interation, which can be apdated to the multi-dimensional case and other nonlinear non-local equations. We develop time decay estimates for solutions of fractional heat equation in mixed norm Lebesgue spaces that could be useful for other problems.

**Key Words**: Local well-posedness, global well-posedness, dissipative quasi-geostrophic equation, fractional heat equation, mixed-norm Lebesgue spaces.

**AMS Subject Classifications**: 35A01, 35K55, 35K61

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## 1 Introduction and main result

We study the Cauchy problem for the 2D dissipative quasi-geostrophic equation

$$\begin{cases} u_t + (-\Delta)^\alpha u = \mathcal{R}(u)\cdot\nabla u & \text{in } \mathbb{R}^2\times(0,T), \\ u(\cdot,0)=\theta_0(\cdot) & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\alpha\in(0,1)$, $u:\mathbb{R}^2\times(0,T)\to\mathbb{R}$ is an unknown solution with some $T>0$, $\theta_0:\mathbb{R}^2\to\mathbb{R}$ is a measurable function of initial data, and

$$\mathcal{R}(u) = (-\mathcal{R}_2(u),\mathcal{R}_1(u))$$

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in which $\mathcal{R}_k$ is the $k$th Riesz transform which is defined by

$$\mathcal{R}_k(u) = \partial_{x_k}(-\Delta)^{-\frac{1}{2}}u, \quad k = 1,2.$$

Moreover, in (1.1), $(-\Delta)^{\alpha}$ denotes the fractional Laplace operator of order $\alpha$ whose precise definition will be recalled in Subsection 2.3.

The goal of this paper is to study the well-posedness of (1.1) in critical mixed-norm Lebesgue spaces. To make sense what we mean by this, let us recall the following scaling invariant property of (1.1). From a simple calculation, we see that for each solution $u$ of (1.1) and each $\lambda > 0$, the rescaled function $u^{\lambda}$ defined by

$$u^{\lambda}(x,t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t), \quad (x,t) \in \mathbb{R}^2 \times (0, T/\lambda^{2\alpha}), \tag{1.2}$$

is also a solution of (1.1) with the corresponding scaled initial data $\theta_0^{\lambda}$ defined as in (1.2).

Now, for each $p_1, p_2 \in (1,\infty)$, the mixed norm $L_{p_1,p_2}(\mathbb{R}^2)$ of a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined in [1] by

$$\|f\|_{L_{p_1,p_2}(\mathbb{R}^2)} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x_1,x_2)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{1}{p_2}}.$$

Similar definitions can be formulated with either or both of $p_1 = \infty$, $p_2 = \infty$. Then, we observe that

$$\|u^{\lambda}(\cdot,t)\|_{L_{p_1,p_2}(\mathbb{R}^2)} = \|u(\cdot,\lambda^{2\alpha}t)\|_{L_{p_1,p_2}(\mathbb{R}^2)} \quad \text{for all } \lambda > 0 \quad \text{and for all } t \in [0, T/\lambda^{2\alpha})$$

if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} = 2\alpha - 1. \tag{1.3}$$

Note that (1.3) is valid only when $\alpha \geq \frac{1}{2}$.

The study of (dissipative) active scalar equations has seen a great topic of research in the last decades, starting with the seminal works of Constantin, Majda and Tabak [7,8]. It is commonly known that (see also (1.3)), Eq. (1.1) is critical for $\alpha = \frac{1}{2}$, subcritical for $\alpha > \frac{1}{2}$ and supercritical otherwise. For the latter, the global well-posedness is largely open (see e.g., [24]). For the subcritical case, the problem has been investigated in [6] (see also e.g., [10,16]) and for the critical case in the seminal work [5] (see the predecessor paper [19] for smooth data and also [9]). The super-critical case has been addressed in [11, 12] where some regularity is assumed for the velocity. It is important to notice that even if in the present work we are considering a subcritical problem as far as the scaling is concerned, the fact that the initial data is chosen in a critical space does not allow to obtain easily a global well-posedness result for large data in our framework. Indeed, the local well-posedness in the critical Lebesgue space $L^{\frac{2}{2\alpha-1}}$ obtained in [6] can be improved to a global one using the $L^p$-maximum principle in [13]. However in our case, we do not