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LIPSCHITZ CONTINUITY OF MINIMIZERS FOR THE GINZBURG-LANDAU FUNCTIONAL BETWEEN ALEXANDROV SPACES*†

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Dedicated to the 90th Birthday of Xiaqi Ding

Abstract

In this paper, we shall prove that any minimizer of Ginzburg-Landau functional from an Alexandrov space with curvature bounded below into a non-positively curved metric cone must be locally Lipschitz continuous.

Keywords Ginzburg-Landau functional, Alexandrov space, NPC metric space

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1 Introduction

Let N be a positive integral and Ω be a bounded domain in a smooth Riemannian manifold. For fixed $K > 0$, recall that the Ginzburg-Landau functional $I_K(\cdot)$ on $W^{1,2}(\Omega, \mathbb{R}^N)$ is defined by

$$I_K(u) := \int_\Omega |\nabla u|^2 + K(1 - |u|^2)^2 \mathrm{d}x. \tag{1.1}$$

This kind of functionals has been originally introduced as a phenomenological phase-field type free-energy of a superconductor. The Ginzburg-Landau functionals have deserved a great attention by the mathematical community too. Starting from the classical monograph [2] (see also [1]) by Bethuel, Brezis and Hélein, many mathematicians have been interested in studying minimization problems for the Ginzburg-Landau energy on smooth manifolds. We refer the readers to, for instance, [1, 2, 27] and references therein for this topic.

In this paper, we will study the Ginzburg-Landau functional from an Alexandrov space with curvature bounded below (shortly, CBB) into a non-positively curved

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(shortly, NPC) metric space. Roughly speaking, an Alexandrov space with CBB is a length space X with the property that any geodesic triangle in X is "fatter" than the corresponding one in the associated model space. The seminal paper [4] and the 10th chapter in the book [5] provide introductions to Alexandrov geometry.

Let us begin from the concept of energy for maps between metric spaces, which was introduced by N. Korevaar and R. Schoen in [23]. Let $(X, |\cdot, \cdot|)$, (Y, d) be two metric spaces and Ω be a bounded domain (connected open subset) of X. μ is a Radon measure on X. Given $p \geq 1$, $\varepsilon > 0$ and a Borel measurable map $u : \Omega \to Y$, the approximating energy functional $E^u_{p,\varepsilon}$ of u is given as follows. For each compactly supported continuous function $\varphi \in C_c(\Omega)$, we set

$$E^u_{p,\varepsilon}(\varphi) := C_{n,p} \int_\Omega \varphi(x) \mathrm{d}\mu(x) \int_{B_x(\varepsilon)\cap\Omega} \frac{d^p(u(x), u(y))}{\varepsilon^{n+p}} \mathrm{d}\mu(y),$$

where $C_{n,p}$ is a normalized constant. The p-th energy functional of u is defined by

$$E^u_p(\varphi) := \limsup_{\varepsilon\to 0} E^u_{p,\varepsilon}(\varphi), \quad \text{for any } \varphi \in C_c(\Omega).$$

We say that $u \in W^{1,p}(\Omega, Y)$ if $u \in L^p(\Omega, Y)$ and it has finite p-energy

$$E^u_p(\Omega) := \sup_{\varphi\in C_c(\Omega),\, 0\leq\varphi\leq 1} E^u_p(\varphi) < \infty.$$

Let (Y, d_Y) be an Euclidean cone with the vertex o and Ω be a domain of X. Given $K > 0$, now one can define the Ginzburg-Landau functional, I_K on $W^{1,2}(\Omega, Y)$ as

$$I_K(u) := E^u_2(\Omega) + \int_\Omega K(1 - |u|^2_o)^2 \mathrm{d}\mu, \tag{1.2}$$

where $|u|_o := d_Y(u, o)$ for any $u \in Y$. We also write $|\cdot|$ instead of $|\cdot|_o$ for short. We say that $u \in W^{1,2}(\Omega, Y)$ is a *minimizer* of (1.2), if for any $v \in W^{1,2}(\Omega, Y)$ with $d(u, v) \in W^{1,2}_0(\Omega)$, we have $I_K(u) \leq I_K(v)$.

The purpose of this paper is to study the regularity theory of minimizers for the Ginzburg-Landau function I_K from a domain of an Alexandrov space with CBB into a complete NPC length space. If $K = 0$, a minimizer of I_K is a harmonic map. In this case, this problem was initiated by F.H. Lin [26] and J. Jost [15-18], independently. They established the locally Hölder continuity for harmonic maps from a domain of an Alexandrov space with CBB into a complete NPC length space. In fact, one can modify their arguments in [17, 26] to deduce the locally Hölder continuity for minimizers of the Ginzburg-Landau function I_K, for any $K > 0$, under the same setting.

Similar as the harmonic maps between singular spaces, an interesting problem is to extend the above Hölder regularity to Lipschitz continuity. In this direction, Serbinowski [35] established the following result.