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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF INTEGRAL BOUNDARY VALUE PROBLEM*†

Yang Yang, Yunrui Yang‡, Kepan Liu
(*School of Math. and Physics, Lanzhou Jiaotong University, Lanzhou 730070, Gansu, PR China*)

Abstract

In this paper, the existence and multiplicity of positive solutions for a class of non-resonant fourth-order integral boundary value problem

$$\begin{cases} u^{(4)}(t)+\beta u''(t)-\alpha u(t)=f\left(t,u(t),u''(t)\right), \quad t\in(0,1),\\ u''(0)=u''(1)=0,\\ u(0)=0, \quad u(1)=\left(\dfrac{1}{\lambda_2}-\dfrac{1}{\lambda_1}\right)\displaystyle\int_0^1 q(s)f(s,u(s),u''(s))\mathrm{d}s \end{cases}$$

with two parameters are established by using the Guo-Krasnoselskii's fixed-point theorem, where $f\in C\left([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty)\right)$, $q(t)\in L^1[0,1]$ is nonnegative, $\alpha,\beta\in R$ and satisfy $\beta<2\pi^2$, $\alpha>0$, $\alpha/\pi^4+\beta/\pi^2<1$, $\lambda_{1,2}=(-\beta\mp\sqrt{\beta^2+4\alpha})/2$. The corresponding examples are raised to demonstrate the results we obtained.

Keywords positive solutions; fixed point; integral boundary conditions
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1 Introduction

By the fact of wide applications in a number of scientific fields of boundary value problems for ordinary differential equations, much attention and discussion has been attracted to many scholars [1–5]. Especially, an increasing interest in the existence and multiplicity of positive solutions to boundary value problems with integral boundary conditions has been evolved recent years, which arises in the fields of thermo-elasticity, heat conduction, plasma physics and underground water. Moreover, this kind of nonlocal boundary value problems include two-point and multi-point cases [3–6]. For more details about boundary value problems with integral boundary conditions, one can refer to [6–9].

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‡Corresponding author. E-mail: lily1979101@163.com

In 2011, Ma [8] considered the existence of positive solutions for the following boundary value problem

$$\begin{cases} u^{(4)}(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u(0) = \displaystyle\int_0^1 g(s)u(s)\mathrm{d}s, \quad u(1) = 0, \\ u''(0) = \displaystyle\int_0^1 h(s)u''(s)\mathrm{d}s, \quad u''(1) = 0 \end{cases}$$

with integral boundary conditions by using the Krein-Rutman theorem and the global bifurcation techniques, where $f \in C\left([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty)\right)$ and $g, h \in L^1[0,1]$ are nonnegative. At the same year, by using the operator spectrum theorem together with the fixed point theorem on cone, Chai [9] established some results on the existence of positive solution for the following integral boundary value problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f\left(t, u(t)\right), \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ u''(0) = \displaystyle\int_0^1 u(s)\phi_1(s)\mathrm{d}s, \quad u''(1) = \displaystyle\int_0^1 u(s)\phi_2(s)\mathrm{d}s \end{cases}$$

with two parameters, where $f \in C\left([0,1] \times [0,+\infty), (-\infty,+\infty)\right)$ is allowed to change sign and $\alpha, \beta \in R$, $\beta < 2\pi^2$, $\alpha \geq -\beta^2/4$, $\alpha/\pi^4 + \beta/\pi^2 < 1$, $\phi_1, \phi_2 \in C\left([0,1], (-\infty,0]\right)$.

The above results mentioned are mainly dealt with the existence of positive solutions. However, there are not few results on the existence of multiple positive solutions for integral boundary value problems. For completeness, the main purpose of this paper is to investigate the multiplicity of positive solutions for the following non-resonant fourth-order integral boundary value problem

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f\left(t, u(t), u''(t)\right), \quad t \in (0,1), \\ u''(0) = u''(1) = 0, \\ u(0) = 0, \quad u(1) = \left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) \displaystyle\int_0^1 q(s) f(s, u(s), u''(s))\mathrm{d}s \end{cases} \tag{1.1}$$

with two parameters.

Throughout this paper, we make the following assumptions:

(H_1) $\alpha, \beta \in R$, $\beta < 2\pi^2$, $\alpha > 0$, $\alpha/\pi^4 + \beta/\pi^2 < 1$;

(H_2) λ_1, λ_2 are the two roots of the polynomial $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$ and $\lambda_{1,2} = (-\beta \mp \sqrt{\beta^2 + 4\alpha})/2$;

(H_3) $f \in C\left([0,1] \times [0,+\infty) \times (-\infty,0], [0,+\infty)\right)$ and $f(t,u,v) > 0$, for any $t \in [\frac{1}{4}, \frac{3}{4}]$ and $|u| + |v| > 0$; $q(t) \in L^1[0,1]$ are nonnegative satisfying that there exists a number $M > 0$ such that