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UNIFORM STABILITY AND ERROR ANALYSIS FOR SOME DISCONTINUOUS GALERKIN METHODS *

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Abstract

In this paper, we provide a number of new estimates on the stability and convergence of both hybrid discontinuous Galerkin (HDG) and weak Galerkin (WG) methods. By using the standard Brezzi theory on mixed methods, we carefully define appropriate norms for the various discretization variables and then establish that the stability and error estimates hold uniformly with respect to stabilization and discretization parameters. As a result, by taking appropriate limit of the stabilization parameters, we show that the HDG method converges to a primal conforming method and the WG method converges to a mixed conforming method.

Mathematics subject classification: 65N30.
Key words: Uniform Stability, Uniform Error Estimate, Hybrid Discontinuous Galerkin, Weak Galerkin.

1. Introduction

In the last few decades, one variant of finite element method called the discontinuous Galerkin (DG) method [1, 2] has been developed to solve various differential equations due to their flexibility in constructing feasible local shape-function spaces and the advantage of effectively capturing non-smooth or oscillatory solutions. Since DG methods use discontinuous space as trial space, the number of degrees of freedom is usually much higher than the standard conforming method. To reduce the number of globally coupled degrees of freedom of DG methods, a hybrid DG (HDG) has been developed. The idea of hybrid methods can be tracked to the 1960s [3]. A new hybridization approach in [4] was put forward by Cockburn and Gopalakrishnan in 2004 and was successfully applied to a discontinuous Galerkin method in [5]. Using the local discontinuous Galerkin (LDG) method to define the local solvers, a superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems was designed in [6]. In 2009, a unified analysis for the hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second-order elliptic problems was presented in [7] by Cockburn, Gopalakrishnan, and Lazarov. A projection-based error analysis of HDG methods was presented in [8], where a projection was constructed to obtain the L^2 error estimate for the potential and flux. However, the error estimate was dependent on the stabilization parameter.

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A projection-based analysis of the hybridized discontinuous Galerkin methods for convection-diffusion equations for semi-matching nonconforming meshes was presented in [9]. An analysis for a hybridized discontinuous Galerkin method with reduced stabilization for second-order elliptic problem was given in [10].

Based on a new concept, namely the weak gradient, introduced in [11], Wang and Ye proposed a weak Galerkin (WG) method for elliptic equations. Similar to the concept introduced in [11], Wang and Ye [12] introduced a concept called weak divergence. Based on the newly introduced concept, Wang and Ye [12] proposed and analyzed a WG method for the second-order elliptic equation formulated as a system of two first-order linear equations. Then a similar idea was applied to Darcy-Stokes flow in [13]. A primal-dual WG finite element method for second-order elliptic equations in non-divergence form was presented in [14] and a further similar method was applied to Fokker-Planck type equations in [15]. A bridge building the connection between the WG method and HDG method was shown in [16]. A summary of the idea and applications of WG methods to various problem were provided in [17].

In this paper, in contrast to the projection-based error analysis in [8, 10], we use the Ladyzhenskaya-Babuška-Brezzi (LBB) theory to prove two types of uniform stability results under some carefully constructed parameter-dependent norms for HDG methods. Based on the uniform stability results, we prove uniform and optimal error estimates for HDG methods. In addition, by using properly defined parameter-dependent norms, we further prove two types of uniform stability results for WG methods. Similarly based on the uniform stability results, we provide uniform and optimal error estimates for WG methods. These uniform stability results and error estimates for WG methods are meaningful and interesting improvement for the results in [11, 12]. Following these uniform stability results for HDG methods and WG methods presented in this paper, an HDG method is shown to converge to a primal conforming method, whereas a WG method is shown to converge to a mixed conforming method by taking the limit of the stabilization parameters.

We illustrate the main idea and results by using the following elliptic boundary value problem:

$$-\mathrm{div}(\alpha\nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain and $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and symmetric positive definite matrix, and its inverse is denoted by $c = \alpha^{-1}$. Setting $\boldsymbol{p} = -\alpha\nabla u$, the above problem can be written as:

$$c\boldsymbol{p} + \nabla u = 0, \quad -\mathrm{div}\boldsymbol{p} = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

The rest of the paper is organized as follows. In Section 2, some preliminary materials are provided. In Section 3, we set up the HDG and WG methods and provide the main uniform well-posedness results. Based on the uniform well-posedness results, we present uniform and optimal error estimates for HDG and WG in Section 4, and show that an HDG method converges to a primal conforming method, whereas a WG method converges to a mixed conforming method by taking the limit of the stabilization parameters in Section 5. In Section 6, we provide proof of the uniform well-posedness of HDG and WG under the specific parameter-dependent norms. We provide a brief summary in the last section.