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Finite-Time Blow-Up and Local Existence for Chemotaxis System with a General Memory Term

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Abstract. In this paper, we discuss the local existence of weak solutions for a parabolic system modelling chemotaxis with memory term, and we show the finite-time blow-up and chemotactic collapse for this system. The main methods we used are the fixed point theorem and the semigroup theory.

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Key Words: parabolic system; chemotaxis; memory term; fixed point theorem; blow-up.

1 Introduction

In this paper, we consider the following model:

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v) + \int_0^t f[u(\cdot,\tau)]\mathrm{d}\tau & \text{in } \Omega\times(0,T), \\ v_t = \Delta v - v + u & \text{in } \Omega\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega\times\{0\}, \\ v(\cdot,0) = v_0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

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where $\Omega\subset\mathbb{R}^N$, a bounded open domain with smooth boundary $\partial\Omega$, $\vec{n}$ is the unit outer normal on $\partial\Omega$ and χ is a nonnegative constant. And f is a continuous linear function and it satisfies the condition: $\|f(x_1)-f(x_2)\|\leq L\|x_1-x_2\|$, where L is a positive constant.

Our model is initiated by the PKS model which is a mathematical model of biological phenomena. And this model for chemosensitive movement has been developed by Patlak, Keller and Segel [1].

$$\begin{cases} u_t=\nabla(\nabla u-\chi(v)u\nabla v), \\ \varepsilon v_t=\Delta v+g(u,v), \end{cases} \tag{1.2}$$

where u represents the population density and v denotes the density of the external stimulus, χ is the sensitive coefficient, the time constant ε $(0\leq\varepsilon\leq1)$ indicates that the spatial spread of the organisms u and the signal v are on different time scales. The case $\varepsilon=0$ corresponds to a quasi-steady state assumption for the signal distribution.

Since the PKS model is designed to describe the behavior of bacteria and bacteria aggregates, the question arises whether or not this model is able to show aggregation. Plenty of theoretical research uncovered exact conditions for aggregations and for blow up (see, e.g., Childress and Percus [2, 3], Jäger and Luckhaus [4], Nagai [5], Gajewski et al. [6], Senba [7], Rasde and Ziti [8], Herrera and Velasquez [9], Othmer and Stevens [10] or Levine and Sleeman [11]).

Global existence below these thresholds has been proven using a Lyapunov functional in Gajewski, et al. [6], Nagai, et al. [12] and Biler [13]. Besides, a number of theoretical research found exact conditions for aggregations and other properties [14–16]. Free boundary problems for the chemotaxis model are considered [17–20].

Our study of (1.1) is also motivated by the following problem for the heat equation with a general time integral boundary condition [21]:

$$\begin{cases} u_t=\Delta u & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial\vec{n}}=\displaystyle\int_0^t f[u(x,s)]\mathrm{d}s & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x) & x\in\overline{\Omega}, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$with boundary $\partial\Omega\subset C^{1+\mu}(0<\mu<1)$, $\vec{n}$ is the outward normal, and $u_0(x)$ is a nonnegative function such that

$$\frac{\partial u_0}{\partial\vec{n}}=0 \quad \text{for } x\in\partial\Omega,$$

f is a nondecreasing function with $f\in C^1(0,\infty)$ and $f(0)>0$.

Considering the nonlinear time integral condition governing flux through the boundary, the model (1.1) involves a continuous time delay which is often referred to as a memory condition in the literature. This memory term can perfectly describe the movement