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Existence of Solutions to Elliptic Equations with Variable Exponents and a Singular Term

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Abstract: The purpose of this paper is to study a class of elliptic equations with variable exponents. By using the method of regularization and a priori estimates, we obtain the existence of weak solutions to these problems.

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1 Introduction

In this paper, we focus on the existence of solutions to the following quasi-linear elliptic equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \dfrac{f(x)}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $f \geq, \not\equiv 0$, $\alpha \geq 1$, $p(x)$ is continuous in Ω satisfying

$$1 < p^- = \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) = p^+ < \infty, \tag{1.2}$$

and the following logarithmic module of continuity condition

$$|p(z) - p(\xi)| \leqslant \omega(|z-\xi|), \qquad z, \xi \in \Omega,\ |z-\xi| < 1, \tag{1.3}$$

where $\limsup\limits_{\tau\to 0^+} \omega(\tau)\ln\dfrac{1}{\tau} = C < +\infty$.

The $p(x)$-Laplace equation is naturally arisen in physical phenomenon, which could be used to describe the non-linear heat-exchanging problem, inhomogeneous material motion

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and image processing problem (see [1]–[4]). When $p = 2$, the equation has been completely solved. When $f \in C^\beta$ $(0 < \beta < 1)$, Lazer and Mckenna[5] declared that there exists a solution in $H_0^1(\Omega)$ if and only if $\alpha < 3$, while for $\alpha > 1$, the solution of the equation does not belong to $C^1(\bar{\Omega})$. Boccardo and Orsina[6] discussed how the integrability of f and the value of α influence the existence, regularity and non-existence of the weak solutions. After Lazer's results, Lair and Shaker[7] proved the existence of weak and classical solutions for semi-linear elliptic equations with general singular terms.

As to the quasi-linear case, Giacomoni *et al.*[8] considered the following equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \dfrac{\lambda}{u^\alpha} + u^q, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha < 1$, $p - 1 < q \le p^* - 1$ (p^* is the Sobolev conjugate exponent of p) and showed the existence and the multiplicity of the positive solutions. Later, Loc and Schmitt[9] improved the results under weaker assumptions.

Motivated by the works mentioned above, we consider the existence of solutions to the problem (1.1). It should be pointed out that the upper and lower solution method used in the previous works is not applicable to $p(x)$-Laplace equations since it is very difficult to construct proper upper and lower solutions. On the other hand, in general there is no classical solution for the equation (1.1) because of the degeneracy and singularity of the equation. In this paper, we consider the existence of weak solutions. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1) if for any $\Omega' \subset\subset \Omega$, there exists a $C_{\Omega'}$ such that $u \ge C_{\Omega'} > 0$ in Ω' and

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \mathrm{d}x = \int_\Omega \frac{f\varphi}{u^\alpha}\mathrm{d}x, \qquad \varphi \in W_0^{1,p(x)}(\Omega). \tag{1.4}$$

To show the existence of such solutions, we combine the regularization method with a priori estimates to overcome the difficulties caused by the $p(x)$-Laplace operator and the singular term.

2 Main Result

We define some notations and introduce some elementary lemmas before stating and proving our main results, and the interested readers may refer to [10] for the proofs.

We denote by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions on Ω such that

$$A_{p(\cdot)}(u) \equiv \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < \infty.$$

This is a Banach space with respect to the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x < 1\right\}$$

for any $u \in L^{p(\cdot)}(\Omega)$. If $p \in C(\bar{\Omega})$, $C^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. By $W^{1,p(x)}(\Omega)$ we denote the space of functions such that $u \in L^{p(x)}(\Omega)$ and $u_{x_i} \in L^{p(x)}(\Omega)$ $(i = 1, 2, \cdots, N)$ equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{1,p(x),\Omega} = \|u\|_{p(x),\Omega} + \|\nabla u\|_{p(x),\Omega}.$$