J. Math. Study
doi: 10.4208/jms.v54n3.21.03

Vol. **54**, No. 3, pp. 262-336
September 2021

On Function Spaces with Mixed Norms — A Survey

Long Huang and Dachun Yang*

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.

Received 9 August 2019; Accepted 12 November 2019

Published online 29 March 2021

Abstract. The targets of this article are threefold. The first one is to give a survey on the recent developments of function spaces with mixed norms, including mixed Lebesgue spaces, iterated weak Lebesgue spaces, weak mixed-norm Lebesgue spaces and mixed Morrey spaces as well as anisotropic mixed-norm Hardy spaces. The second one is to provide a detailed proof for a useful inequality about mixed Lebesgue norms and the Hardy–Littlewood maximal operator and also to improve some known results on the maximal function characterizations of anisotropic mixed-norm Hardy spaces and the boundedness of Calderón–Zygmund operators from these anisotropic mixed-norm Hardy spaces to themselves or to mixed Lebesgue spaces. The last one is to correct some errors and seal some gaps existing in the known articles.

AMS subject classifications: 42B35, 42B30, 42B25, 42B20

Key words: Mixed norm, (weak) Lebesgue space, Morrey space, Hardy space, maximal function, Littlewood–Paley function, Calderón–Zygmund operator.

1 Introduction

In 1961, the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$, with $\vec{p}\in(0,\infty]^n$, as a natural generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$ via replacing the constant exponent p by an exponent vector $\vec{p}$, was investigated by Benedek and Panzone [9]. Indeed, the origin of these mixed Lebesgue spaces can be traced back to the interesting article of Hörmander [44] on the estimates for translation invariant operators, in 1960. Later on, in 1965, Galmarino and Panzone [35] extended the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ to the mixed Lebesgue space $L^P(\mathbb{R}^n)$ with the exponent P being a sequence, namely, an ∞-tuple. Since the early 1960s, lots of nice work have been done in the study of the boundedness of operators on mixed norm spaces; see, for instance, Benedek *et al.* [8], Lizorkin [64], Adams and Bagby [1],

*Corresponding author. *Email addresses:* `longhuang@mail.bnu.edu.cn` (L. Huang), `dcyang@bnu.edu.cn` (D. Yang)

Schmeisser [74], Rubio de Francia *et al.* [73] and Fernandez [33] as well as Stefanov and Torres [76]. Recently, the Plancherel–Polya inequality on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ and the wavelet characterization of $L^{\vec{p}}(\mathbb{R}^n)$ were studied by Torres and Ward [84]; the smoothing properties of bilinear operators and Leibniz-type rules in mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ were considered by Hart *et al.* [41]; the boundedness of the multilinear strong maximal operator from the product of mixed Lebesgue spaces to mixed Lebesgue spaces was obtained by Liu *et al.* [61]. In addition, more recently, Córdoba and Latorre Crespo in [26] revisited some classical conjectures in harmonic analysis in the setting of mixed norm spaces. To be exact, they established the sharp boundedness for the restriction of the Fourier transform to compact hypersurfaces of revolution and studied an extension of the disc multiplier in the mixed norm setting. For more progresses about the mixed Lebesgue space, we refer the reader to [3, 4, 19, 20, 27, 43, 48, 49, 65, 75].

On another hand, motivated by the aforementioned work of Benedek and Panzone [9] on mixed Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, numerous other function spaces with mixed norms were introduced and studied. For instance, Besov spaces, Sobolev spaces and Bessel potential spaces with mixed norms were investigated by Besov *et al.* [11, 12] in the 1970s; inhomogeneous Triebel–Lizorkin spaces with mixed norms were also studied by Besov *et al.* in [13]; parabolic function spaces with mixed norms were considered by Gopala Rao [39]. Particularly, in 1977, Fernandez [32] first introduced the Lorentz spaces with mixed norms. Later, an interpolation result on these Lorentz spaces with mixed norms was obtained by Milman [69]. Moreover, Lorentz–Marcinkiewicz spaces with mixed norms and Orlicz spaces with mixed norms were considered by Milman in [67] and [68], respectively; Banach function spaces with mixed norms were studied by Blozinski [14]; anisotropic mixed-norm Hardy spaces were introduced by Clenathous *et al.* [23]; mixed-norm α-modulation spaces were researched by Cleanthous and Georgiadis [21]; mixed Lebesgue spaces with variable exponents were considered by Ho [43]; Morrey spaces with mixed norms were investigated by Nogayama [71, 72]; mixed martingale Hardy spaces were studied by Szarvas and Weisz [81]. Indeed, the function spaces with mixed norms have attracted considerable attention and have rapidly been developed. For more progresses about various function spaces with mixed norms and their applications in the boundedness of different operators, we refer the reader to [19,22,24,25,36,37,41,42,51–54].

In the last two decades, due to the wider usefulness of function spaces with mixed norms within the context of partial differential equations, there has been a renewed interest in the study of them. More precisely, since the function spaces with mixed norms have finer structures than the corresponding classical function spaces, they naturally arise in the studies on the solutions of partial differential equations used to model physical processes involving in both space and time variables, such as the heat or the wave equations (particularly, the very useful Strichartz estimates); see, for instance, [3, 56–58, 83]. This is also based on the fact that, while treating some linear or nonlinear equations, functions with different orders of integrability in different variables give more precise information on the parameters involved in the estimates and further induce a better regularity (of traces) of solutions; see, for instance, [28, 38, 85]. Another recent interest in developing