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Weighted and Maximally Hypoelliptic Estimates for the Fokker-Planck Operator with Electromagnetic Fields

Wei-Xi Li[1,*] and Juan Zeng[2]

[1] *School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, China.*
[2] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.*

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Abstract. We consider a Fokker-Planck operator with electric potential and electromagnetic fields. We establish the sharp weighted and subelliptic estimates, involving the control of the derivatives of electric potential and electromagnetic fields. Our proof relies on a localization argument as well as a careful calculation on commutators.

AMS subject classifications: 35H20, 35Q84

Key words: Maximal estimate, global hypoellipticity, Fokker-Planck operator.

1 Introduction and main results

There have been several works on the Fokker-Planck operator with electric potential $V(x)$ which is

$$K=y\cdot\partial_x-\partial_x V(x)\cdot\partial_y-\triangle_y+\frac{|y|^2}{4}-\frac{n}{2},\quad (x,y)\in\mathbb{R}^{2n}, \tag{1.1}$$

where x denotes the space variable and y denotes the velocity variable, and $V(x)$ is a potential defined in the whole space $\mathbb{R}^n_x$. It is a degenerate operator with

*Corresponding author. *Email addresses:* wei-xi.li@whu.edu.cn (W.-X. Li), juan-zeng@whu.edu.cn (J. Zeng)

the absence of diffusion in x variable, and can be seen as a Kolmogorov-type operator. The classical hypoelliptic techniques and their global counterparts have been developed recently to establish global estimates and to investigate the short and long time behavior and the spectral properties for Fokker-Plack operator K in (1.1). We refer to Helffer-Nier's notes [4] for the comprehensive argument on this topic, seeing also the earlier work [6] of Hérau-Nier. In the first author's work [11,12] we improved the previous result and gave a new criterion involving the microlocal property of potential V. Here we also mention the very recent progress made by Ben Said-Nier-Viola [2] and Ben Said [1]. Finally as a result of the global estimates it enables to answer partially a conjecture stated by Helffer-Nier [4] which says Fokker-Planck operator K has a compact resolvent if and only if Witten Laplacian has a compact resolvent. The necessity part is well-known and the reverse implication still remains open with some partial answers; in fact various hypoelliptic techniques, such as Kohn' method and nilpotent approach (e.g., [5, 9, 14]), were developed to establish the resolvent criteria for these two different type operators (see [4, 11–13]).

Inspired by the recent work of Helffer-Karaki [3], we consider here a more general Fokker-Planck operator with electromagnetic fields besides the electric potential, which reads

$$P=y\cdot\partial_x-\partial_x V(x)\cdot\partial_y-H(x)\cdot(y\wedge\partial_y)-\triangle_y+\frac{|y|^2}{4}-\frac{n}{2},\quad (x,y)\in\mathbb{R}^{2n}, \tag{1.2}$$

where $n=2$ or 3 and $H(x)$ is a scalar function of x for $n=2$ and a vector field $(H_1(x),H_2(x),H_3(x))$ of only x-variable for $n=3$, and $y\wedge\partial_y$ is defined by

$$y\wedge\partial_y=\begin{cases} y_1\partial_{y_2}-y_2\partial_{y_1}, & n=2,\\ \big(y_2\partial_{y_3}-y_3\partial_{y_2},\, y_3\partial_{y_1}-y_1\partial_{y_3},\, y_1\partial_{y_2}-y_2\partial_{y_1}\big), & n=3. \end{cases}$$

The operator is initiated by Helffer-Karaki [3], where they established the maximal estimate by virtue of nilpotent approach, giving a criteria for the compactness of the resolvent. Here we aim to give another proof, basing on a localization argument and a careful calculation on commutators. Note the operator P in (1.2) is reduced to the operator K given (1.1) for $H\equiv 0$; meanwhile the maximal estimates for the Fokker-Planck operator with pure electromagnetic fields (i.e., $V\equiv 0$) was investigated by Zeinab Karaki [8].

Before stating our main result we first introduce some notations used throughout the paper. We will use $\|\cdot\|_{L^2}$ to denote the norm of the complex Hilbert space $L^2(\mathbb{R}^{2n})$, and denote by $C_0^\infty(\mathbb{R}^{2n})$ the set of smooth compactly supported functions. Denote by $\mathscr{F}_x$ the (partial) Fourier transform with respect to x and