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A New Model of Coupled Hindmarsh-Rose Neurons

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Abstract A new model of two coupled neurons is presented by the partly diffusive Hindmarsh-Rose equations. The solution semiflow exhibits globally absorbing characteristics. As the main result, the self-synchronization of the coupled neurons at a uniform rate is proved, which can be extended to complex neuronal networks.

Keywords Coupled Hindmarsh-Rose equations, Absorbing dynamics, Synchronization of neurons.

MSC(2010) 35B40, 35K57, 35Q80, 37N25, 92C20.

1. Introduction

The Hindmarsh-Rose equations as a mathematical model for neuron firing-bursting were initially proposed in [8]. This model originally composed of three ordinary differential equations has been studied through numerical simulations and bifurcation analysis, cf. [10–12, 18, 20, 22] and the references therein.

In this paper, we present a new model of coupled two neurons in terms of the following system of the coupled partly diffusive Hindmarsh-Rose equations:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= d\Delta u_1 + au_1^2 - bu_1^3 + v_1 - w_1 + J + p(u_2 - u_1),\\
\frac{\partial v_1}{\partial t} &= \alpha - v_1 - \beta u_1^2,\\
\frac{\partial w_1}{\partial t} &= q(u_1 - c) - rw_1,\\
\frac{\partial u_2}{\partial t} &= d\Delta u_2 + au_2^2 - bu_2^3 + v_2 - w_2 + J + p(u_1 - u_2),\\
\frac{\partial v_2}{\partial t} &= \alpha - v_2 - \beta u_2^2,\\
\frac{\partial w_2}{\partial t} &= q(u_2 - c) - rw_2,
\end{aligned}
\tag{1.1}
$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ $(n \leq 3)$, where Ω is a bounded domain with locally Lipschitz continuous boundary. Here (u_i, v_i, w_i), $i = 1, 2$, are the state variables for two Hindmarsh-Rose (HR) neurons.

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The input electrical current $J > 0$ and the coefficient of neuron coupling strength $p > 0$ are treated as constants. For cell biological reason, the coupling terms are only with the two equations of the membrane potential of neuronal cells.

In this system (1.1), the variable $u_i(t,x)$ refers to the membrane electrical potential of a neuronal cell, the variable $v_i(t,x)$ called the spiking variable represents the transport rate of the ions of sodium and potassium through the fast ion channels, and the variable $w_i(t,x)$ called the bursting variable represents the transport rate across the neuronal cell membrane through slow channels of calcium and other ions.

All the involved parameters are positive constants except $c\,(= u_R) \in \mathbb{R}$, which is a reference value of the membrane potential of a neuron cell. In the original ODE model of a single neuron [22], a set of the typical parameters are

$$J = 3.281, \quad r = 0.0021, \quad S = 4.0, \quad q = rS, \quad c = -1.6,$$
$$\varphi(s) = 3.0s^2 - s^3, \quad \psi(s) = 1.0 - 5.0s^2.$$

We impose the homogeneous Neumann boundary conditions for the u_i-components,

$$\frac{\partial u_1}{\partial \nu}(t,x) = 0, \quad \frac{\partial u_2}{\partial \nu}(t,x) = 0, \quad \text{for } t > 0, \ x \in \partial\Omega, \tag{1.2}$$

and the initial conditions to be specified are denoted by $(i = 1, 2)$

$$u_i(0,x) = u_i^0(x), \quad v_i(0,x) = v_i^0(x), \quad w_i(0,x) = w_i^0(x), \quad x \in \Omega. \tag{1.3}$$

The single HR neuron model [8] was motivated by the discovery of neuronal cells in the pond snail *Lymnaea*. This model characterizes the phenomena of synaptic bursting and more interested chaotic bursting in the (u, v, w) space.

Neuronal signals are short electrical pulses called spikes or action potential. Neurons often exhibit bursts of alternating phases of rapid firing spikes and then quiescence. Bursting constitutes a mechanism to modulate and set the pace for brain functionalities and to communicate signals. Synaptic coupling of neurons has to reach certain threshold for release of quantal vesicles and synchronization [5, 15, 17].

The bursting dynamics in chaotic coupling neurons in the simulations and semi-numerical analysis of the Hindmarsh-Rose model in ordinary differential equations exhibited more rapid synchronization and more effective regularization of neurons due to lower threshold than the regular synaptic coupling [20].

Bursting behavior and patterns occur in a variety of excitable cells and bio-systems such as pituitary melanotropic gland, thalamic neurons, respiratory pacemaker neurons, and insulin-secreting pancreatic β-cells, cf. [1, 2, 4, 8]. The mathematical analysis mainly using bifurcations of several models in ODEs on neuron bursting and synchronization has been studied by many authors, cf. [6, 12, 18, 20–22].

It is known that Hodgkin-Huxley equations [9] provided a highly nonlinear four-dimensional model if without simplification. Besides the FitzHugh-Nagumo equations [7] provided a two-dimensional model for an excitable neuron. It admits an exquisite phase plane analysis showing sustained periodic spiking with refractory period, but seems hard to motivate any chaotic solutions and to generate chaotic bursting dynamics.