Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 3, September 2020, 333–343 DOI:10.12150/jnma.2020.333

Note on Fractional Green's Function*

Jinfa Cheng[1,†] and Yuming Chu[2]

Abstract In this paper, we modify some errors on the definition of fractional Green's function in monograph [5], and give the solution of the inhomogenous equation which satisfies the given inhomogenous initial conditions by fractional Green's function.

Keywords Fractional Green's function, Laplace transform, Mittage-Leffler function.

MSC(2010) 26A33, 65L05.

1. Introduction

In the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in science and engineering, such as control, porous media, electrochemistry, viscoelasticity, and electromagnetism theory [1-3, 5-6], ect. There are a large number of papers dealing with the fractional differential equations [7-11]. The Fractional Green's function is a very powerful tool for investigating linear fractional differential equations [3-6]. In Chapter 5 of the monograph [5], the fractional Green's function is defined as follows:

Consider the following equation

$$ {}_0\mathscr{L}_t y(t) \equiv f(t), \tag{1.1}$$

$$[{}_0\mathscr{D}_t^{\sigma_k - 1} y(t)]_{t=0} = 0, (k = 1, \cdots, n),$$

where

$${}_a\mathscr{L}_t y(t) \equiv {}_a \mathscr{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t) {}_a\mathscr{D}_t^{\sigma_{n-k}} y(t) + p_n(t) y(t),$$

$${}_a\mathscr{D}_t^{\sigma_k} y(t) \equiv {}_a D_t^{\alpha_k} {}_a D_t^{\alpha_{k-1}} \cdots {}_a D_t^{\alpha_1},$$

$${}_a\mathscr{D}_t^{\sigma_k - 1} y(t) \equiv ({}_a D_t^{\alpha_k - 1})({}_a D_t^{\alpha_{k-1}}) \cdots ({}_a D_t^{\alpha_1}),$$

$$\sigma_k = \sum_{j=1}^{k} \alpha_j, (k = 1, 2, \cdots, n); 0 \leq \alpha_j \leq 1, (j = 1, 2, \cdots, n). \tag{1.2}$$

†the corresponding author.
Email address: jfcheng@xmu.edu.cn(J. Cheng), chuyuming@zjhu.edu.cn(Y. Chu)
1School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 316005, China
2Department of Mathematics, Huzhou Teachers College, Huzhou, Zhejiang 313000, China
*The authors were supported by the National Natural Science Foundation Fujian Province of China (2016J01032).

Definition 1.1 (see [5]). The function $G(t,\tau)$ satisfying the following conditions
a) ${}_{\tau}\mathscr{L}_t G(t,\tau) = 0 \quad$ for every $\tau \in (0,t)$;
b) $\lim\limits_{\tau \to t-0} ({}_{\tau}\mathscr{D}_t^{\sigma_k - 1} G(t,\tau)) = \delta_{k,n}, \quad k = 0, 1, \cdots, n,$
($\delta_{k,n}$ is Kronecker's delta);
c) $\lim\limits_{\substack{\tau, t \to +0 \\ \tau < t}} ({}_{\tau}\mathscr{D}_t^{\sigma_k} G(t,\tau)) = 0, k = 0, 1, \cdots, n-1$
is called the fractional Green's function of equation (1.1).

The purpose of our paper is to point out that there exist some errors or contradictions in Definition 1.1, and we will provide some examples to illustrate them and modify them. The paper has been organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. In section 3, we shall point out some errors in Definition 1.1. In section 4 we give some examples to illustrate them and modify Definition 1.1, we also give the solution of the inhomogeneous equation satisfying given inhomogeneous initial conditions by the Laplace transform method and fractional Green's function.

2. Preliminaries and Lemmas

In order to establish our main results we need some preliminary facts and basic lemmas, which we present in this section.

Definition 2.1. Let $f(t)$ be piecewise continuous on $(0,\infty)$ and $p > 0$, then the fractional integral of order p of $f(t)$ is defined by

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. $$

Definition 2.2 (see [1-3, 5-6]). Let $f(t)$ be piecewise continuous on $(0,\infty)$ and $0 \le m-1 \le v < m \in N$, then the Riemann-Liouville fractional derivative of f is defined by

$$ {}_aD_t^{v} f(t) = \frac{d^m}{dt^m} [{}_aD_t^{v-m} f(t)]. $$

In Definitions 2.1 and 2.2, if $a = 0$, then we denote ${}_aD_t^{-p} f(t)$ and ${}_aD_t^{v} f(t)$ by $D^{-p} f(t)$ and $D^{v} f(t)$, respectively.

Lemma 2.1 (see [1-3, 5-6]). *Let $\alpha > 0$ and $\beta > 0$, then*

$$ {}_aD_t^{-\alpha}[(t-a)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, $$

$$ {}_aD_t^{\alpha}[(t-a)^{\beta-1}] = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}. $$

Lemma 2.2. (see [5-6]). *If $0 \le m-1 \le v < m \in N$, and σ_m is defined as in (1.2), then we have two classical formulas for the Laplaces transform of the fractional derivative as follows:*

$$ L[{}_0D_t{}^{v} f(t)] = s^v F(s) - \sum_{k=0}^{m-1} s^{m-k-1} [{}_0D_t^{k-m+v} f(t)]_{t=0} $$