Journal of Nonlinear Modeling and Analysis
Volume 2, Number 3, September 2020, 345–353
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.345

New Peakons and Periodic Peakons of the Modified Camassa-Holm Equation*

Xinhui Lu[1], Lin Lu[2,†] and Aiyong Chen[1,2]

Abstract In this paper, we obtain new peakon and periodic peakon solutions to a modified Camassa-Holm equation. We change the modified Camassa-Holm equation into a planar system. Then the first integral and algebraic curves of this system are obtained. By using the first integral and algebraic curves, a new peakon solution is given by hyperbolic function. Moreover, some new periodic peakons are given by elliptic functions and triangle functions.

Keywords Camassa-Holm equation, Peakon, Periodic peakon, Solitary wave, Periodic wave.

MSC(2010) 34C25, 34C60, 37C27.

1. Introduction

In recent years, studies on Camassa-Holm equations have received considerable attention because these equations have many applications in physics. The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was proposed by Camassa and Holm [1] as a model equations for shallow water unidirectional nonlinear dispersion waves, where $u(x,t)$ representing the waters free surface over a flat bed. Equation (1.1) admits the peakons and periodic peakons in the following forms [1,5]

$$u(x,t) = ce^{-|x-ct|}$$

and

$$u(x,t) = \frac{c}{\sinh(1/2)} \cosh\left(\frac{1}{2} - (x-ct) + [x-ct]\right),$$

where the notation $[x]$ denotes the largest integer part of the real number $x \in \mathbb{R}$. Lenells [4] obtained smooth solitary wave solutions to the famous Korteweg-de Vries equation

$$u_t - 6uu_x + uu_{xxx} = 0, \tag{1.2}$$

†the corresponding author.
Email address: xiaoxiaolui@163.com(X. Lu), 14726989141@163.com(L. Lu), aiyongchen@163.com(A. Chen)
[1]School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin, Guangxi 541004, China
[2]Department of Mathematics, Hunan First Normal University, Changsha, Hunan 410205, China
*The authors were supported by the National Natural Science Foundation of China (No.11671107), Guangxi Natural Science Foundation of China (No.2015GXNSFGA139004), Hunan Provincial Natural Science Foundation of China (No.2017JJ3044) and Scientific Research Fund of Hunan Provincial Education Department (No.17C0333).

and showed that the smooth traveling waves of (1.1) naturally correspond to traveling waves of (1.2).

The μ-Camassa-Holm (μCH) equation

$$\mu(u_t) - u_{xxt} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx} \tag{1.3}$$

was introduced as an integrable equation arising in the study of the diffeomorphism group of the circle, where $u(x,t)$ is a real-valued spatially periodic function and $\mu(u) = \int_{S^1} u(x,t)dx$ denotes its mean. Lenells et al. [6] obtained that equation (1.3) admits periodic peakons

$$u(x,t) = \frac{c}{26}\left(12(x-ct)^2 + 23\right),$$

where $|x - ct| \le \frac{1}{2}$ and $u(x,t)$ is extended periodically to the real line.

In this paper, we consider the modified Camassa-Holm (mCH) equation

$$u_t - u_{xxt} = u u_{xxx} + 2u_x u_{xx} - 3u^2 u_x, \tag{1.4}$$

where $x \in \mathbb{R}$ and $t > 0$. Wazwaz [12] obtained some solitary wave solutions to (1.4) by using the extended tanh method and the rational hyperbolic functions method. Moreover, based on the method of complete discrimination system for polynomial, Deng [2] obtained some exact travelling wave solutions to (1.4). Since the nonlinear partial differential equations have various traveling wave solutions [3,7–11], inspired by the above, the aim of this paper is to construct new peakons and periodic peakons solutions by using first integral and algebraic curves. A peakon solution is given by a hyperbolic function, and some new periodic peakons are given by elliptic functions and triangle functions.

The rest of the paper is organized as follows. In Section 2, we change (1.4) into a planar system. Then we obtain the first integral of the planar system, and use Maple to draw the bifurcation of each algebraic curve on the phase plane. In Section 3, we obtain new peakons and periodic peakons solutions to equation (1.4).

2. First integral and algebraic curve

By substituting $u(x,t) = \phi(\xi)$ with $\xi = x - ct$ into equation (1.4), it follows that

$$-c\phi' + c\phi''' = \phi\phi''' + 2\phi'\phi'' - 3\phi^2\phi', \tag{2.1}$$

where ϕ' is the derivative with respect to ξ. Integrating equation (2.1) once, we obtain

$$(\phi - c)\,\phi'' + \frac{1}{2}(\phi')^2 - \phi^3 + c\phi = g, \tag{2.2}$$

where g is the integral constant.

Letting $y = \frac{d\phi}{d\xi}$, then we obtain the following planar dynamic system

$$\begin{cases} \dfrac{d\phi}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{-\frac{1}{2}y^2 + \phi^3 - c\phi + g}{\phi - c}. \end{cases} \tag{2.3}$$