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On the Growth in Time of Sobolev Norms for Time Dependent Linear Generalized KdV-type Equations*

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Abstract We give a detailed description in 1-D the growth of Sobolev norms for time dependent linear generalized KdV-type equations on the circle. For most initial data, the growth of Sobolev norms is polynomial in time for fixed analytic potential with admissible growth. If the initial data are given in a fixed smaller function space with more strict admissible growth conditions for $V(x,t)$, then the growth of previous Sobolev norms is at most logarithmic in time.

Keywords Sobolev norms, Time dependent linear generalized KdV-type equation, Fixed analytic potential.

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1. Introduction

When consider the growth in time of Sobolev norms for nonlinear Hamiltonian partial differential equations (PDEs), we can choose the linearized equations of these PDEs to study first. The main example discussed before is the nonlinear Schrödinger equation

$$\mathrm{i}u_t = \Delta u + \frac{\partial H}{\partial \bar{u}}, x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \tag{1.1}$$

with Hamiltionian

$$H = \int_{\mathbb{T}} \left(|\nabla u|^2 + F(|u|^2)\right) dx, \tag{1.2}$$

where F is a polynomial or smooth function. Let $\phi(x) = u(0,x)$ be the initial data. For given $J \gg 1$, split the data ϕ in low and high Fourier modes as

$$\phi = \phi_1 + \phi_2, \tag{1.3}$$

where

$$\phi_1 = \Pi_J \phi = \sum_{|j| \le J} \hat{\phi}(j) \mathrm{e}^{\mathrm{i}jx}. \tag{1.4}$$

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Write

$$u = w + v. \tag{1.5}$$

Then w and v satisfy, respectively, the initial value problems

$$\begin{cases} \mathrm{i}\partial_t w = \Delta w + \frac{\partial H}{\partial \bar{w}} \\ w(0) = \phi_1 \end{cases} \tag{1.6}$$

and

$$\begin{cases} \mathrm{i}\partial_t v = \Delta v + \frac{\partial^2 H}{\partial \bar{w} \partial w} v + \frac{\partial^2 H}{\partial w^2} \bar{v} + F(w, \bar{w}, v, \bar{v}) \\ v(0) = \phi_2. \end{cases} \tag{1.7}$$

It can be turned out that $F(w, \bar{w}, v, \bar{v})$ is a high order term expected to have a small effect and $(\mathrm{i}\partial_t + \Delta)^{-1}$ has a smoothing effect on the term $\frac{\partial^2 H}{\partial \bar{w}^2} \cdot \bar{v}$. Finally we can use the fact that the flow of the linear equation

$$\begin{cases} \mathrm{i}\partial_t v = \Delta v + \frac{\partial^2 H}{\partial \bar{w} \partial w} v \\ v(0) = \phi_2 \end{cases} \tag{1.8}$$

conserves the L^2- norm and has essentially unitary behavior in H^s (up to lower-order error terms) since $\frac{\partial^2 H}{\partial \bar{w} \partial w}$ is real. Therefore, it is reasonable to investigate the growth in time of Sobolev norms for the linearized equation firstly. In the appendix of [2], using Floquet theory Bourgain proved that the Sobolev norm of the linear Schrödinger equation of the form

$$\mathrm{i}u_t + \Delta u + V(x,t)u = 0 \tag{1.9}$$

with periodic boundary conditions satisfies polynomial growth. And Wang obtained the result of logarithmic growth of Sobolev norm for the equation (1.9) in 2008 [6]. Also see [3–5] and the references therein. Essentially, it is there proved that in a period of time, the H^s norm of the high frequencies part is preserved. Besides, using localization properties of eigenfunction, the approximate solution can be constructed by Floquet solution, and the middle frequencies part is controlled.

For other nonlinear Hamiltonian PDEs, in 1996 [1], Bourgain mentioned the Sobolev norm growth of the generalized KdV-type equations in the periodic case of the form

$$u_t + u_{xxx} + \partial_x f'(u) = 0, \tag{1.10}$$

where f is sufficiently smooth. To that end, here we firstly investigate the Sobolev norm growth of the linearised KdV-type equation,

$$u_t + u_{xxx} + \frac{1}{2} V_x(x,t)u + V(x,t)u_x = 0, x \in \mathbb{T}, \tag{1.11}$$

where the potential $V(x,t)$ is time-dependent, bounded and real. We further assume that $V(x,t)$ is real analytic in (x,t) in a strip $D := (\mathbb{R} + \mathrm{i}\rho)^2 (|\rho| < \rho_0, \rho_0 > 0)$. Here $\frac{1}{2}$ is to guarantee that the flow of (1.11) conserves the L^2- norm.

Since the nonlinear term $\frac{1}{2} V_x(x,t)u + V(x,t)u_x$ has derivative term u_x, it leads to complication in the proof of polynomial growth and the control of high frequencies part. Hence, some admissible growth conditions for $V(x,t)$ are necessary. What is more, if we want to obtain logarithmic growth, then the initial data must be given