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Some New Exact Solutions for Time Fractional Thin-film Equation

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Abstract In this paper, the invariant subspace method is utilized to obtain some new exact solutions for the time fractional thin-film equation. The fractional derivative in the considered equation is given in Remain-Liouville and Caputo senses. Some new invariant subspaces have been obtained that are not reported in the literature before.

Keywords Time-fractional thin-film equation, Riemann-Lioville fractional derivative, Caputo fractional derivative, Invariant subspace method, New exact solutions.

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1. Introduction

Exact solutions of nonlinear evolution equations play a very important role in the study of nonlinear physical phenomena. Many methods can be utilized for obtaining exact solutions of nonlinear evolution equations such as Backlund transformation method [6, 9], Lie group method [1, 3, 19, 24], the tanh method [2, 5, 22, 29], the $\exp(-\varphi(z))-$expansion method [12, 13, 15, 17], the exp function approach [16] and the invariant subspace method (ISM) [7, 21].

The importance of the invariant subspace method comes from it is not only used for solving nonlinear evolution equations but also it can be used for solving fractional nonlinear evolution equations. Very recently, it is widely utilized for investigating exact solutions of fractional nonlinear evolution equations (see for example [4, 8, 10, 11, 14, 18, 20, 26–28, 30]).

In this paper, we use the ISM to investigate some new solutions of the time-fractional thin-film equation [27]

$$\frac{\partial^\alpha u}{\partial t^\alpha}= -u\left(\frac{\partial^4 u}{\partial x^4}\right)+\beta\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^3 u}{\partial x^3}\right)+\gamma\left(\frac{\partial^2 u}{\partial x^2}\right)^2 , \qquad t> 0\ ,\ 0 <\alpha\leq 1\ . \tag{1.1}$$

Equation (1.1) can be used as a model for thin film flow on a substrate [7] . Here, $u(x,t)$ denotes the height of the fluid. The invariant subspaces and some exact solutions of the Eq. (1.1) (when $\alpha = 1$) have been obtained in [7, 30]. Some exact solution of Eq. (1.1) have been obtained in [27] using the ISM. The main aim of

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this paper is to obtain some new invariant subspaces and some new exact solutions of Eq. (1.1).
The rest of the paper is organized as follows: The basics and definitions of the ISM are introduced in Section 2. The new inavariant subspaces and exact solutions of Eq. (1.1) are discussed in Section 3. Section 4 discuss the results and conclusion of the paper.

2. ISM: Time Fractional Partial Differential Equations (PDEs)

Consider a time fractional PDE

$$\frac{\partial^\alpha u}{\partial t^\alpha}=F\left[u\right]=F\left(x,t,u,u_x,u_{xx},\ldots\ldots\right),\qquad 0<\alpha\leq 1, \tag{2.1}$$

where $F\left[u\right]$ is a nonlinear differential operator of order k. The n-dimensional invariant subspace $W_n=\mathcal{L}\left\{f_1\left(x\right),f_2\left(x\right),\ldots,f_k\left(x\right)\right\}$ (where $n\leq 2k+1$) can be obtained from the solution of the following nth-order linear ordinary differential equation:

$$L\left[y\right]=y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_0y\left(x\right)=0, \tag{2.2}$$

where, the constant coefficients $a_{n-1},\ a_{n-2},\ldots,a_0$ can be obtained from the condition [13]

$$L\left[F\left[u\right]\right]|_{L[u]=0}=0.$$

The time fractional PDE (2.1) can be converted into a system of time fractional ordinary differential equations (ODEs) through the following theorem:

Theorem 1 [13]**.** Let W_n be the linear space spanned by n linearly independent functions $\{\ f_i(x),i\ =\ 1,\ 2,...,\ n\}$ and suppose that W_n is invariant under the operator $F\left[u\right]$, which means that

$$F\left[\sum_{i=1}^{n}c_if_i\left(x\right)\right]=\sum_{i=1}^{n}\overline{F}_i\left(c_1,c_2,\ldots.c_n\right)f_i\left(x\right), \tag{2.3}$$

for whatever constants $c_1,c_2,\ldots,c_n$. Then, the fractional PDE (2.1) has the solution of the form

$$u(x,t)=\sum_{i=1}^{n}c_i\left(t\right)f_i\left(x\right), \tag{2.4}$$

where the coefficients $c_1(t),c_2(t),\ldots.c_n(\mathrm{t})$ satisfy the following system of fractional ODEs

$$\frac{d^\alpha c_i(t)}{dt^\alpha}=\psi_i\ \left(c_1\left(t\right),\ldots.,c_n\left(\mathrm{t}\right)\right),\qquad i\ =\ 1,\ 2,...,\ n. \tag{2.5}$$

Here, the fractional order derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be considered in the Riemann-Liouville sense in some cases. In some other cases the fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be in the Caputo sense according to the availability of exact solutions of the obtained system of fractional ODEs after using the invariant subspace method . Basic definitions and properties of the Riemann- Liouville and Caputo fractional derivatives are given in Appendix A.