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Dynamics of the Stochastic Chemostat Model with Monod-Haldane Response Function*

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Abstract This paper is devoted to the asymptotic dynamics of stochastic chemostat model with Monod-Haldane response function. We first prove the existence of random attractors by means of the conjugacy method and further construct a general condition for internal structure of the random attractor, implying extinction of the species even with small noise. Moreover, we show that the attractors of Wong-Zakai approximations converges to the attractor of the stochastic chemostat model in an appropriate sense.

Keywords Stochastic chemostat model, random attractors, Wong-Zakai approximation, conjugacy method.

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1. Introduction

Chemostat refers to a basic piece of laboratory apparatus used for the continuous culture of microorganisms. It occupies a central place in mathematical ecology and has played an important role in many fields [4, 12, 16, 22, 30–32, 34]. It can also model waste water treatment [13, 26] or study recombinant problems in genetically altered microorganisms [17, 18]. Derivation and analysis of chemostat models are well documented in [9, 29, 33] and references therein.

The classic chemostat model with single species and single limiting substrate takes the form

$$\frac{dS(t)}{dt} = (S^0 - S(t))D - \mu(S(t))x(t), \tag{1.1}$$

$$\frac{dx(t)}{dt} = -Dx(t) + \mu(S(t))x(t), \tag{1.2}$$

where $S(t)$ and $x(t)$ denote concentrations of the nutrient and the microbial biomass, respectively; S^0 denotes the volumetric dilution rate and D is the dilution rate. The

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growth rate of the microbial population is represented by the function $\mu(S)$, which is generally assumed to be non-negative.

However, there are very strong restrictions as the real world is non-autonomous and stochastic, and this justifies the analysis of stochastic chemostat model. In general, there exist several alternatives to model randomness and stochasticity. For example, one can replace the dilution rate D by $D + \alpha\dot{W}(t)$ and thus the original system (1.1)-(1.2) is replaced by the following stochastic differential equations understood in the Itô sense

$$dS(t) = [(S^0 - S(t))D - \mu(S(t))x(t)]dt + \alpha(S^0 - S(t))dW(t), \tag{1.3}$$

$$dx(t) = [-Dx(t) + \mu(S(t))x(t)]dt - \alpha x(t)dW(t), \tag{1.4}$$

where $W(t)$ is a standard Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, and $\alpha > 0$ is its intensity. Biologically the model does not seem completely realistic due to the fact that the substrate $S(t)$ in the corresponding stochastic chemostat model (1.3)-(1.4) can take negative. Alternatively, following the idea in [15, 19], one can obtain the stochastic chemostat model

$$dS(t) = [(S^0 - S(t))D - \mu(S(t))x(t)]dt - \alpha S(t)dW(t), \tag{1.5}$$

$$dx(t) = [-Dx(t) + \mu(S(t))x(t)]dt - \alpha x(t)dW(t). \tag{1.6}$$

Recently, the existence of the random attractor associated to the random dynamical system generated by the solution of system (1.3)-(1.4) (or (1.5)-(1.6)) was studied in [5–7] by using a function Holling type-II, $\mu(S(t)) = mS(t)/(a+S(t))$, where a is the half-saturation constant and m is the maximal consumption rate of the nutrient and also the maximal specific growth rate of microorganisms. In particular, authors in [6] proved the existence of the global random attractor of system (1.5)-(1.6) with Holling type-II respond function, and further shown that the internal structure of the attractor consists of singleton subsets as long as $\bar{D} = D + \alpha^2/2 > m$, which means that the microorganisms become extinct. In fact, one can choose α, large enough, such that $\bar{D} > m$ (see Figure 2 in [6]). In case $\bar{D} < m$, one cannot ensure the persistence of the microorganism (see Figure 1 in [6]).

As far as we know, no report has been found on the existence of random attractors of stochastic chemostat model under small noise. This fact inspires us to further explore relevant dynamics of system (1.5)-(1.6) in this respect. Besides, some experiments and observations indicate that not only insufficient nutrient but also excessive nutrient may inhibit the growth of a microbial population in the chemostat [1, 3, 20]. This situation suggested a non-monotonic response function, so-called Monod-Haldane function, to model such growth. Thus system (1.5)-(1.6) becomes the following specified form

$$dS(t) = [(S^0 - S(t))D - \frac{mS(t)x(t)}{a + S(t) + KS^2(t)}]dt - \alpha S(t)dW(t), \tag{1.7}$$

$$dx(t) = [-Dx(t) + \frac{mS(t)x(t)}{a + S(t) + KS^2(t)}]dt - \alpha x(t)dW(t), \tag{1.8}$$

where the term $KS^2(t)$ describes the inhibitory effect of the substrate at high concentrations. By using the well-known conversion between Itô and Stratonovich senses, we obtain the following stochastic chemostat with Monod-Haldane function

$$dS(t) = [-\bar{D}S(t) - \frac{mS(t)x(t)}{a + S(t) + KS^2(t)} + S^0 D]dt - \alpha S(t) \circ dW(t), \tag{1.9}$$