Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 1, Number 3, September 2019, 397–413 DOI:10.12150/jnma.2019.397

# Lyapunov-type Inequalities for Fractional $(p, q)$-Laplacian Systems*

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**Abstract** In this paper, we establish some new Lyapunov type inequalities for fractional $(p, q)$-Laplacian operators in an open bounded set $\Omega \subset \mathbb{R}^N$, under homogeneous Dirichlet boundary conditions. Next, we use the obtained inequalities to derive some geometric properties of the generalized spectrum associated to the considered problem.

**Keywords** Fractional Sobolev spaces, fractional $(p, q)$-Laplacian operators, Lyapunov inequality, generalized eigenvalues.

**MSC(2010)** 35P15, 35P30.

## 1. Introduction

The Lyapunov inequality and its various generalizations have found applications in the study of properties of solutions such as oscillation theory, asymptotic theory, eigenvalue problems of differential and difference equations. On the other hand, the fractional p-Laplacian operator is a class of non-local pseudo differential operators. The equations involving the fractional p-Laplacian operators are used to describe the diffusion phenomenon, which has been widely used in fluid mechanics, material memory, biology, plasma physics, finance, and so on. In the last few decades, many authors have established various Lyapunov type inequalities for fractional p-Laplacian operators, see, for example the Refs. [2–6] and the references therein.

In [4], Mohamed Jleli, Mokhtar Kirane and Bessem Samet considered the fractional p-Laplacian operator $(-\Delta_p)^s$, where $1 < p < \infty$, $s \in (0, 1)$, in an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, under homogeneous Dirichlet boundary conditions. More precisely, they considered the following problem

$$\begin{cases} (-\Delta_p)^s u = w|u|^{p-2}u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega, \end{cases}$$

where the weight function $w \in L^\infty(\Omega)$. They discussed two cases, the case $sp > N$ and the case $sp < N$. For each case, they obtained a Lyapunov-type inequality involved the inner radius of the domain and $L^\theta$ norms of the weight $w$.

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* Q. Li is supported by National Natural Science Foundation of China (No 11571090).

In [5], Mohamed Jleli and Bessem Samet considered the following system involving $(p_i,\, q_i)$-Laplacian operators $(i = 1,\, 2)$:

$$\begin{cases} -(|u'(x)|^{p_1-2}u'(x))' - (|u'(x)|^{q_1-2}u'(x))' = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), \\ -(|v'(x)|^{p_2-2}v'(x))' - (|v'(x)|^{q_2-2}v'(x))' = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), \end{cases} \tag{1.1}$$

on the interval $(a,\, b)$, under Dirichlet boundary conditions

$$u(a) = u(b) = v(a) = v(b) = 0.$$

System (1.1) is investigated under the assumptions

$$\alpha \geq 2, \;\; \beta \geq 2, \;\; p_i \geq 2, \;\; q_i \geq 2, \;\; (i = 1,\, 2),$$

and

$$\frac{2\alpha}{p_1 + q_1} + \frac{2\beta}{p_2 + q_2} = 1.$$

Where $f$ and $g$ are two nonnegative real-valued functions such that $(f,\, g) \in L^1(a,\, b) \times L^1(a,\, b)$. It was proved that if (1.1) has a nontrivial solution $(u,\, v) \in C^2[a,\, b] \times C^2[a,\, b]$, then

$$\Big[\min\{\frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}}\}\Big]^{\frac{2\alpha}{p_1+q_1}} \Big[\min\{\frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}}\}\Big]^{\frac{2\beta}{p_2+q_2}} \\ \leq \Big(\frac{1}{2}\int_a^b f(x)dx\Big)^{\frac{2\alpha}{p_1+q_1}} \Big(\frac{1}{2}\int_a^b g(x)dx\Big)^{\frac{2\beta}{p_2+q_2}}.$$

Some nice applications to generalized eigenvalues are also presented in [5].

In this paper, we establish some new Lyapunov type inequalities for fractional Laplacian systems. More precisely, we consider:

$$\begin{cases} (-\Delta_{p_1})^s u(x) + (-\Delta_{p_2})^s u(x) = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}u(x), & \\ (-\Delta_{q_1})^s v(x) + (-\Delta_{q_2})^s v(x) = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}v(x), & \text{in} \quad \Omega, \\ u = v = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.2}$$

System (1.2) is investigated under the assumptions

$$s \in (0,\, 1), \;\; \alpha \geq 2, \;\; \beta \geq 2, \;\; p_i \geq 2, \;\; q_i \geq 2, \;\; (i = 1,\, 2),$$

and

$$\frac{2\alpha}{p_1 + p_2} + \frac{2\beta}{q_1 + q_2} = 1. \tag{1.3}$$

We also consider the system:

$$\begin{cases} \sum_{i=1}^{3}[(-\Delta_{p_i})^s u(x)] = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta}|w(x)|^{\gamma}u(x), & \\ \sum_{i=1}^{3}[(-\Delta_{q_i})^s v(x)] = g(x)|u(x)|^{\alpha}|v(x)|^{\beta-2}|w(x)|^{\gamma}v(x), & \\ \sum_{i=1}^{3}[(-\Delta_{r_i})^s w(x)] = h(x)|u(x)|^{\alpha}|v(x)|^{\beta}|w(x)|^{\gamma-2}w(x), & \text{in} \quad \Omega, \\ u = v = w = 0, & \text{on} \quad \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.4}$$