Journal of Nonlinear Modeling and Analysis
Volume 1, Number 2, June 2019, 167–177
Website:http://jnma.ca
DOI:10.12150/jnma.2019.167

Periodic Solutions of a Class of Duffing Differential Equations*

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Abstract In this work we study the existence of new periodic solutions for the well knwon class of Duffing differential equation of the form $x'' + cx' + a(t)x + b(t)x^3 = h(t)$, where c is a real parameter, $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. Our results are proved using three different results on the averaging theory of first order.

Keywords Periodic solution, averaging method, Duffing differential equation, bifurcation, stability.

MSC(2010) 34C15, 34C25.

1. Introduction and statement of the main result

Several classes of Duffing differential equations have been investigated by many authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability and bifurcation. See the survey of J. Mawhin [12] and for example the articles [2–4, 6, 9, 10, 13, 16, 18, 19].

In this work we shall study the class of Duffing differential equations of the form

$$x'' + cx' + a(t)x + b(t)x^3 = h(t), \tag{1.1}$$

where $c > 0$ is a constant, and $a(t)$, $b(t)$ and $h(t)$ are continuous T–periodic functions. These differential equations were studied by Chen and Li in the papers [2,3]. These authors studied the periodic solutions of equation (1.1) with the following additional conditions, either $b(t) > 0$, $h(t) > 0$ and $a(t)$ satisfies

$$a(t) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}, \quad \text{and} \quad a_0 = \frac{1}{T}\int_0^T a(t)dt > 0; \tag{1.2}$$

or $a(t) = a > 0$, $b(t) = 1$ and $c > 0$, a, c constants.

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*This work is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

In [8] the authors studied the existence and the stability of periodic solutions of the Duffing differential equation (1.1) with $c = \varepsilon C > 0$, $a(t) = \varepsilon^2 A(t)$, $A_0 b_0 > 0$ where $A_0 = \dfrac{1}{T}\displaystyle\int_0^T A(t)dt$ and $b_0 = \dfrac{1}{T}\displaystyle\int_0^T b(t)dt$, and ε is sufficiently small.

Instead of working with the Duffing differential equation (1.1) we shall work with the equivalent differential system

$$\begin{aligned} x' &= y, \\ y' &= -cy - a(t)x - b(t)x^3 + h(t). \end{aligned} \tag{1.3}$$

We define the polynomial

$$\begin{aligned} p(x_0) = &-\left(\int_0^T e^{-ct}\int_0^t e^{cs}b(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs}b(s)\,ds\right)x_0^3 \\ &-\left(\int_0^T e^{-ct}\int_0^t e^{cs}a(s)\,ds\,dt - \frac{e^{-cT}}{c}\int_0^T e^{cs}a(s)\,ds\right)x_0 \\ &+\frac{e^{-cT}}{c}\int_0^T e^{cs}h(s)\,ds + \int_0^T e^{-ct}\int_0^t e^{cs}h(s)\,ds\,dt. \end{aligned}$$

Our first result on the periodic solutions of the differential system (1.3) is the following.

Theorem 1.1. *For every simple real root of the polynomial* $p(x_0)$ *the differential system* (1.3) *has a periodic solution* $(x(t), y(t))$ *such that* $(x(0), y(0)) = (x_0, 0)$.

Theorem 1.1 will be proved in section 3 using Theorem 4.1 of the averaging theory.

Now we define the polynomial

$$q(x_0) = -\left(\int_0^T b(s)\,ds\right)x_0^3 - \left(\int_0^T a(s)\,ds\right)x_0 + \int_0^T h(s)\,ds.$$

Theorem 1.2. *For every simple real root of the polynomial* $q(x_0)$ *the differential system* (1.3) *has a periodic solution* $(x(t), y(t))$ *such that* $(x(0), y(0)) = (0, 0)$.

Theorem 1.2 will be proved in section 4 using Theorem 4.2 of the averaging theory.

As we shall see Theorem 4.3 of the averaging theory will provide results on the periodic solutions of system (1.3) which are already contained in Theorems 1.1 and 1.2.

In order to apply the three theorems of the averaging theory of first order for studying the periodic solutions of the differential system (1.3) in section 2 we rescale the variables, the parameters and the functions of system (1.3).

The results of averaging theory that we use in this paper are described in section 4.