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Periodic Solutions of the Duffing Differential Equation Revisited via the Averaging Theory*

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Abstract We use three different results of the averaging theory of first order for studying the existence of new periodic solutions in the two Duffing differential equations $\ddot{y} + a\sin y = b\sin t$ and $\ddot{y} + ay - cy^3 = b\sin t$, where a, b and c are real parameters.

Keywords Periodic solution, averaging method, Duffing differential equation, bifurcation, stability.

MSC(2010) 34C15, 34C25.

1. Introduction and statement of the main results

Hamel [6] in 1922 gaves the first general results for the existence of periodic solutions of the periodically forced pendulum equation

$$\ddot{y} + a\sin y = b\sin t, \tag{1.1}$$

where the dot denotes derivative with respect to the independent variable t , also called the time, and $y \in \mathbb{S}^1$ is an angle. Four years earlier this equation was the main subject of a monograph published by Duffing [4], who restricted his study to the periodic solutions of the following approximate equation

$$\ddot{y} + ay - cy^3 = b\sin t. \tag{1.2}$$

This equation is now known as the *Duffing differential equation*. The differential equation (1.2) describes the motion of a damped oscillator with a more complicated potential than in the harmonic motion (i.e. when $c = 0$). As usual the parameter a controls the size of stiffness, b controls the amplitude of the periodic driving force, and c controls the amount of nonlinearity in the restoring force. In particular, equation (1.2) models a spring pendulum such that its spring's stiffness only obey approximately the Hooke's law.

Many other different classes of Duffing differential equations have been investigated by several authors. They are mainly interested in the existence of periodic solutions, in their multiplicity, stability, bifurcation, ... See for instance the good survey of Mawhin [10] and for example the articles [2, 3, 5, 7, 8, 12, 14, 17, 18].

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In this work we shall study the periodic solutions of the Duffing differential equations (1.1) and (1.2), where a, b and c are real parameters, via the averaging theory.

Our main results on the periodic solutions of the Duffing differential equation (1.1) are the following.

Theorem 1.1. *Let ε be a small parameter. The Duffing differential equation* (1.1) *has*

(a) *four periodic solutions $y_1(t) = -b\sin t + O(\varepsilon)$, $y_2(t) = \pi - b\sin t + O(\varepsilon)$, $y_3(t) = O(\varepsilon)$, $y_4(t) = \pi + O(\varepsilon)$ if $ab \neq 0$, $a = O(\varepsilon^2)$ and $b = O(\varepsilon)$;*

(b) *two periodic solutions $y_i(t)$ for $i = 3, 4$ if $b = 0$ and $a \neq 0$;*

(c) *infinitely many periodic solutions $y(t) = k - b\sin t$ with $k \in \mathbb{R}$ if $a = 0$ and $b \neq 0$;*

(d) *no periodic solutions if $a = b = 0$.*

Theorem 1.2. *The generalized eigenfunction systems $v_0^0(x), v_0^1(x)$, $\{u_m^0(x)\}_{m=1}^{+\infty}$ and $\{u_m^1(x)\}_{m=1}^{+\infty}$ of the operator F are complete in the sense of Cauchy principal value in Z .*

Theorem 1.1 will be proved in section 2 using the averaging theorems given in the Appendix.

We note that Theorem 1.1 provides new results with respect to Theorem 2.1 of Tarantello [16] where the author provides conditions for having zero, one or two periodic solutions, while in Theorem 1.1 we provide conditions for having zero, two, four or infinitely many periodic solutions.

Our main results on the periodic solutions of the differential system (1.2) are the following.

Theorem 1.3. *The Duffing differential equation* (1.2) *has*

(a) *one periodic solution $y(t) = -\sqrt[3]{4b/(3c)}\sin t + O(\varepsilon)$, if $bc \neq 0$, $a = 1$ and $b = O(\varepsilon)$ and $c = O(\varepsilon)$;*

(b) *one periodic solution $y(t) = O(\varepsilon)$, if $b \neq 0$, $a = O(\varepsilon)$, $b = O(\varepsilon)$ and $c = O(\varepsilon^2)$;*

(c) *two periodic solutions $y_\pm(t) = \pm\sqrt{a/c}$, if $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon)$ and $c = O(\varepsilon^2)$;*

(d) *three periodic solutions $y(t) = y_0$, where $y_0 \in \{0, \pm\sqrt{a/c}\}$ if $ac > 0$, $b \neq 0$, $a = O(\varepsilon^2)$, $b = O(\varepsilon^2)$ and $c = O(\varepsilon^2)$.*

Theorem 1.3 will be proved in section 3 using three different averaging theorems.

2. Proof of Theorem 1.1

Instead of working with the Duffing differential equation (1.1) we shall work with the equivalent differential systems

$$\begin{aligned} \dot{x} &= -a\sin y + b\sin t, \\ \dot{y} &= x. \end{aligned} \tag{2.1}$$