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Numerical Method for Homoclinic and Heteroclinic Orbits of Neuron Models

Bo Deng[1,†]

Abstract A twisted heteroclinic cycle was proved to exist more than twenty-five years ago for the reaction-diffusion FitzHugh-Nagumo equations in their traveling wave moving frame. The result implies the existence of infinitely many traveling front waves and infinitely many traveling back waves for the system. However efforts to numerically render the twisted cycle were not fruitful for the main reason that such orbits are structurally unstable. Presented here is a bisectional search method for the primary types of traveling wave solutions for the type of bistable reaction-diffusion systems the FitzHugh-Nagumo equations represent. The algorithm converges at a geometric rate and the wave speed can be approximated to significant precision in principle. The method is then applied for a recently obtained axon model with the conclusion that twisted heteroclinic cycle maybe more of a theoretical artifact.

Keywords FitzHugh-Nagumo equations, twisted heteroclinic loop bifurcation, singular perturbation, bisection method.

MSC(2010) 34, 35, 37, 65, 92.

1. Introduction

The reaction-diffusion equations

$$\begin{cases} v_t = v_{xx} + f(v) - w \\ w_t = \epsilon(v - \gamma w) \end{cases} \tag{1.1}$$

with $f(v) = v(v-a)(1-v),\ 0 < a < 1/2,\ \gamma > 0,\ \epsilon > 0,\ t \geq 0,\ x \in \mathbb{R}$ was proposed by FitzHugh [1]and Nagumo [2] as a model for action potential impulses traveling along nerve axon. Researchers have been interested in the type of impulses which can be approximated by a fixed pulse profile but traveling at a constant speed. Such is a solution of one variable in the so-called moving frame: $(v, w)(t, x) = (V, W)(x + ct)$. Denote by $X' = \frac{dX}{dz}$ with $z = x + ct$. Then, $V(z), W(z)$ satisfy the following first order system of ordinary differential equations:

$$\begin{cases} V' = U \\ U' = cU - f(V) + W \\ W' = \frac{\epsilon}{c}(V - \gamma W) \end{cases} \tag{1.2}$$

†the corresponding author.
Email address:bdeng@math.unl.edu
[1]Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA

A traveling impulse is a homoclinic orbit of the origin $(V, U, W) = \mathbf{0} = (0, 0, 0)$, satisfying

$$\lim_{z\to-\infty}(V, U, W)(z) = \mathbf{0} \quad \text{and} \quad \lim_{z\to\infty}(V, U, W)(z) = \mathbf{0}.$$

Most early studies in the literature were about such homoclinic solutions, c.f. [3–7].

When parameter γ is sufficiently large $\gamma > v_{max}/f_{max}$ (with $f_{max} = f(v_{max})$ and $v_{max} = [(a+1) + \sqrt{(a+1)^2 - 3a}]/3$ being the local maximal value of function f), the system has three equilibrium steady states which are on the plane $U = 0$, and on each branch of the cubic nonlinearity $W = f(V)$, separated by the critical points of the curve. The first is the origin $\mathbf{0}$ from which the impulse solution originates; the second is between the local minimal point v_{min} and the local maximal point v_{max} along the V axis which will not appear in our consideration from now on; and the third is above the local maximal point, denoted by $\mathbf{p} = (V_p, 0, W_p)$ with (V_p, W_p), being the solution to the equilibrium equation $W = f(V)$, $V = \gamma W$, which together with $\mathbf{0}$ are the focus of a few studies, including this one, in the literature. When the system is in this so-called bistable configuration, it is possible to have a front wave solution, corresponding to a heteroclinic orbit from $\mathbf{0}$ to $\mathbf{p}$:

$$\lim_{z\to-\infty}(V, U, W)(z) = \mathbf{0} \quad \text{and} \quad \lim_{z\to\infty}(V, U, W)(z) = \mathbf{p}.$$

Similarly a back wave solution may also exist, corresponding to a heteroclinic orbit from $\mathbf{p}$ to $\mathbf{0}$:

$$\lim_{z\to-\infty}(V, U, W)(z) = \mathbf{p} \quad \text{and} \quad \lim_{z\to\infty}(V, U, W)(z) = \mathbf{0}.$$

Traveling front and back waves were considered in [8–12]. In particular, Rinzel and Terman [9] studied the existence of a heteroclinic loop and its bifurcation to homoclinic orbits of both $\mathbf{0}$ and $\mathbf{p}$ for a closely related model to the FN equations.

The work of [13, 14] was the first to consider the effect of the orientation of the global stable manifolds in relation to the heteroclinic loop on the bifurcations of heteroclinic orbits other than the primary types forming the loop. The following result was obtained in Deng [14, 15]:

Theorem 1.1 (Theorem 1.1, [14], Theorem 2.3, [15])**.** *For each $0 < a < 1/2$ there is a sufficiently small $\epsilon_0 > 0$ and a fixed neighborhood $\mathcal{N} = (\gamma_1, \gamma_2) \times (c_1, c_2)$ of the point $\mathcal{B}_0 := (\gamma_0, c_0) = ((1 - 2a)/\sqrt{2}, 9/(2 - a)(1 - 2a))$ so that the following statements hold for every $0 < \epsilon < \epsilon_0$ and in $\mathcal{N}$ for system (1.2):*

1. *There are continuously differentiable curves $c = c_{f,0}(\gamma, \epsilon)$, $c = c_{b,0}(\gamma, \epsilon)$ satisfying at the singular limit $\epsilon = 0$:*

$$c_{f,0}(\gamma, 0) = (1 - 2a)/\sqrt{2}, \quad c = c_{b,0}(\gamma, 0) = -(\beta_1 + \beta_3 - 2\beta_2)/\sqrt{2}$$

for every $\gamma_1 < \gamma < \gamma_2$, where $\beta_1 < \beta_2 < \beta_3$ are the roots to the polynomial equation $-f(v) + W_p = 0$. Moreover, the two curves intersect at a point $(\gamma_\epsilon, c_\epsilon)$ with the properties that $(\gamma_\epsilon, c_\epsilon) = \mathcal{B}_0$ at $\epsilon = 0$, and the curve $c_{b,0}(\gamma, \epsilon)$ decreases through $c_{f,0}(\gamma, \epsilon)$ in γ at γ_ϵ.

2. *For each $\gamma_1 < \gamma < \gamma_2$, the system has a simple front wave at speed $c_{f,0}(\gamma, \epsilon)$ and a simple back wave at speed $c_{b,0}(\gamma, \epsilon)$. In particular, at $(\gamma_\epsilon, c_\epsilon)$ there is a heteroclinic loop from $\mathbf{0}$ to $\mathbf{p}$ and back to $\mathbf{0}$.*