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Dynamics and Bifurcation Study on an Extended Lorenz System

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Abstract In this paper, we study dynamics and bifurcation of limit cycles in a recently developed new chaotic system, called extended Lorenz system. A complete analysis is provided for the existence of limit cycles bifurcating from Hopf critical points. The system has three equilibrium solutions: a zero one at the origin and two non-zero ones at two symmetric points. It is shown that the system can either have one limit cycle around the origin, or three limit cycles enclosing each of the two symmetric equilibria, giving a total six limit cycles. It is not possible for the system to have limit cycles simultaneously bifurcating from all the three equilibria. Simulations are given to verify the analytical predictions.

Keywords Lorenz system, extended Lorenz system, Hopf bifurcation, limit cycle, normal form.

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1. Introduction

Bifurcation of limit cycles has been extensively studied in planar vector fields, for example see the review article [10] and books [4,5], as well as the references therein. This problem is closely related to the well-known Hilbert's 16th problem [6]. The second part of Hilbert's 16th problem is to find an upper bound on the number of limit cycles that planar polynomial systems can have. This number is called Hilbert number, denoted by $H(n)$, where n is the degree of the polynomials. A modern version of this problem was later formulated by Smale, chosen as one of his 18 most challenging mathematical problems for the 21st century [20]. So far, for quadratic systems, the best result is $H(2) \geq 4$, obtained almost 40 years ago [1, 18, 19], but $H(2) = 4$ is still open. For cubic systems, many results have been obtained on the lower bound of $H(3)$, with the best result obtained so far as $H(3) \geq 13$ [9, 11].

On the other hand, Hopf bifurcation [7] has been applied for considering bifurcation of limit cycles from equilibria for a long time, for example see [15], and many

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physical examples can be found, for example in [5]. Recently, particular attention was paid to some 3-dimensional dynamical systems, e.g., see [2, 12, 26]. On the other hand, in the past two decades, many chaotic systems have been constructed to study chaos control and chaos synchronization. However, in order to achieve the two goals, the dynamics of the system such as stability and bifurcation must be explored. Among those systems, the family of Lorenz system [13] is an important class of chaotic systems to be studied. Thus, investigating systems which are similar to the Lorenz family yet not equivalent to Lorenz family is certainly very interesting and of importance for theoretical studies. In fact, a so-called extended Lorenz system was developed in [16] to study chaotic dynamics, which is described by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= m\,x - n\,y - m\,xz - p\,x^3, \\ \dot{z} &= -\,a\,z + b\,x^2, \end{aligned} \tag{1.1}$$

where m, n, p, a and b are real parameters. It can be shown that using the following transformation:

$$\begin{aligned} &x = \sqrt{2}\,X, \quad y = \sqrt{2}\big(X + \tfrac{Y}{\sigma}\big), \quad z = \tfrac{X^2}{\sigma} + (\rho - 1)\,Z, \\ &a = -\,r, \quad b = \tfrac{2\sigma - r}{\sigma(\rho-1)}, \quad m = \sigma(\rho - 1), \quad p = 1, \quad n = \sigma + 1, \end{aligned} \tag{1.2}$$

system (1.1) can be transformed to the standard Lorenz system:

$$\begin{aligned} \dot{X} &= \sigma(Y - X), \\ \dot{Y} &= \rho\,X - Y - XZ, \\ \dot{Z} &= -\,r\,Z + XY. \end{aligned} \tag{1.3}$$

However, it should be noted in (1.2) that p is particularly taken as 1, and so in general the two systems (1.1) and (1.3) are not topologically equivalent. In fact, we will see in the next section that system (1.1) can have more limit cycles than that system (1.3) can. Therefore, in this paper, particular attention is paid to the original system (1.1). Note that both the extended Lorenz system (1.1) and the classical Lorenz system (1.3) are invariant under the transformation: $(x, y, z) \to (-x, -y, z)$. Hence, solution trajectories of these two systems are symmetric with the z-axis. Also, note that the z-axis (or Z-axis for the classical Lorenz system) is an invariant manifold. Trajectories starting from the z-axis (or Z-axis for the classical Lorenz system) either converges to the origin if $a > 0$ or diverges to $\pm\infty$ if $a < 0$.

Hopf bifurcation for the classical Lorenz system (1.3) has been studied by many authors. In particular, Pade *et al.* [17] applied the center manifold theory and averaging method to prove that the Hopf bifurcation is subcritical for $r > 0$ and $\sigma > r + 1$. This implies that for the classical Lorenz system, only two limit cycles (with 1-1 distribution) can bifurcate from the two symmetric equilibria, which contradicts the existence of six limit cycles (with 3-3 distribution) shown in [23]. Hopf bifurcation was also considered for the Chen [8, 14] and Lü [27, 28] chaotic systems (which belong to the so called Lorenz family), but no attention was paid to multiple limit cycle bifurcations in these articles.