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A Remark on Adaptive Decomposition for Nonlinear Time-frequency Analysis

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Abstract: In recent study the bank of real square integrable functions that have nonlinear phases and admit a well-behaved Hilbert transform has been constructed for adaptive representation of nonlinear signals. We first show in this paper that the available basic functions are adequate for establishing an ideal adaptive decomposition algorithm. However, we also point out that the best approximation algorithm, which is a common strategy in decomposing a function into a sum of functions in a prescribed class of basis functions, should not be considered as a candidate for the ideal algorithm.

Key words: Hilbert transform, empirical mode decomposition, adaptive decomposition algorithm, best approximation

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1 Introduction

An important objective for the time-frequency analysis of a signal is to obtain a time-frequency-energy distribution representing at each time its frequencies and the energy corresponding to each frequency. Instantaneous amplitude and frequency are basic concepts in the implementation of this objective.

A classical way of defining without ambiguity the instantaneous amplitude and frequency of a real signal $f \in L^2(\mathbf{R})$ is through the Hilbert transform, which is defined for each function $g \in L^p(\mathbf{R})$, $1 \le p < \infty$, at $x \in \mathbf{R}$ as

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$$(Hg)(x) := \text{p.v. } \frac{1}{\pi}\int_{\mathbf{R}} \frac{g(y)}{x-y}\mathrm{d}y := \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|y-x|\geq\varepsilon}\frac{g(y)}{x-y}\mathrm{d}y,$$

if the Cauchy principal value (p.v. for short) of the above singular integral exists (see [1]). To see this, we form the analytic signal Af of f by letting

$$Af := f + \mathrm{i}Hf$$

and then write Af as

$$(Af)(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}, \qquad t \in \mathbf{R},$$

where $\rho \geq 0$. If $\theta' \geq 0$, then we define $\rho(t)$, $\theta'(t)$ as the instantaneous amplitude and frequency of the signal f at time t, respectively. The quantity $\rho(t)$ is further viewed as the energy of f at time t. To make the above method for defining instantaneous amplitude and frequency applicable, we need an adaptive algorithm $\mathcal{A}$ to decompose an arbitrary real signal into a monotone function and a sum of signals in the following class

$$\mathcal{M} := \{f \in L^2(\mathbf{R}) : f \text{ is real}, \ (Af)(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}, \ \rho \geq 0, \ \theta' \geq 0\}.$$

We also require that the summand in $\mathcal{M}$ for each decomposition decay fast so that the algorithm $\mathcal{A}$ is useful in practice.

In engineering literature, the empirical mode decomposition (EMD) was proposed in [2] to decompose a real signal into a monotone function and a finite sum of functions called intrinsic mode functions (IMFs). An IMF is defined in [2] as a real function ψ with the following two properties:

(a) ψ has exactly one zero between any two consecutive local extrema;

(b) the local mean of ψ is zero.

A basic assumption in the establishment of EMD and the Hilbert-Huang transform (HHT) in [2] is that properties (a) and (b) are an empirical sufficient condition for a signal ψ to belong to $\mathcal{M}$. Along this direction of using the Hilbert transform to define instantaneous amplitude and frequency, the task of building the mathematical foundation for EMD consists of two stages (see [3]). The first is to construct a large bank of functions in $\mathcal{M}$ with explicit expression. The second is to establish the ideal algorithm $\mathcal{A}$ described above.

Results on the construction of functions in $\mathcal{M}$ with explicit expression can be found in [4] and [5]. This study aims at a better understanding of the ideal algorithm $\mathcal{A}$. As a first step, we shall prove in Section 2 that span$\mathcal{M}$ is dense in the space of all the real functions in $L^2(\mathbf{R})$ and show in Section 3 that the best approximation, which is a common strategy in decomposing a function into a sum of functions in a prescribed class of basis functions, should not be considered as a candidate for the algorithm $\mathcal{A}$.

Similar to the real line case, there is a promising method for the time-frequency analysis of periodic but nonlinear signals. The method makes use of the circular Hilbert transform that is defined for each $f \in L^1_{2\pi}$ at $t \in [0, 2\pi]$ as

$$(\tilde{H}f)(t) := \text{p.v. } \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)\cot\frac{s}{2}\mathrm{d}s := \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\varepsilon\leq|s|\leq\pi} f(t-s)\cot\frac{s}{2}\mathrm{d}s,$$

if the Cauchy principal value of the above singular integral exists. Here we use $L^p_{2\pi}$, $1 \leq p \leq \infty$, to denote the set of all the 2π-periodic functions f whose restriction in $[0, 2\pi]$ belongs