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Weak Convergence Theorems for Nonself Mappings

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Abstract: Let E be a real uniformly convex and smooth Banach space, and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let T_1, $T_2 : K \to E$ be two weakly inward nonself asymptotically nonexpansive mappings with respect to P with a sequence $\{k_n^{(i)}\} \subset [1, \infty)$ $(i = 1, 2)$, and $F := F(T_1) \bigcap F(T_2) \neq \emptyset$. An iterative sequence for approximation common fixed points of the two nonself asymptotically nonexpansive mappings is discussed. If E has also a Fréchet differentiable norm or its dual E^* has Kadec-Klee property, then weak convergence theorems are obtained.

Key words: asymptotically nonexpansive nonself-mapping, weak convergence, uniformly convex Banach space, common fixed point, smooth Banach space

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1 Introduction and Preliminaries

Throughout this work, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \ \|f\| = \|x\|\}, \qquad x \in E,$$

where $\langle \cdot , \cdot \rangle$ denotes the duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j. It is well known that if E is a smooth Banach space, then J is single-valued.

A Banach space E is said to have a Fréchet differentiable norm (see [1]), if for all $x \in U = \{x \in E : \|x\| = 1\}$, the limit $\lim\limits_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists and is attained uniformly in

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$y \in U$. In this case there exists an increasing function $b : [0, \infty) \to [0, \infty)$ with $\lim\limits_{t \to 0^+} \dfrac{b(t)}{t} = 0$ such that

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b(\|h\|), \qquad x, h \in E. \tag{1.1}$$

A subset K of E is said to be retract of E if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is retract. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny if $P(Px + t(x - Px)) = Px$ for $x \in C$ with $Px + t(x - Px) \in C$ and $t \ge 0$.

For any $x \in K$, the inward set $I_K(x)$ is defined as follows:

$$I_K(x) = \{y \in E : y = x + \lambda(z - x),\ z \in K,\ \lambda \ge 0\}.$$

A mapping $T : K \to E$ is said to satisfy the inward condition if $Tx \in I_K(x)$ for all $x \in K$. T is said to be weakly inward if $Tx \in \mathrm{cl}I_K(x)$ for each $x \in K$, where $\mathrm{cl}I_K(x)$ is the closure of $I_K(x)$.

A Banach space E is said to have the Kadec-Klee property (see [2]) if for every sequence $\{x_n\}$ in E, with $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ strongly.

We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in K : Tx = x\}$, and by $F := F(T_1) \bigcap F(T_2)$ the set of common fixed points of two mappings T_1 and T_2.

Definition 1.1[3] *Let E be a real normed linear space, and K be a nonempty subset of E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that for any $x, y \in K$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|x - y\|$, $n \ge 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$, $\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le L\|x - y\|$, $n \ge 1$.*

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. Nonself asymptotically nonexpansive mappings have been studied by many authors (see [3–8]). Chidume *et al.*[3] studied the following iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad n \ge 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in (0, 1), and proved some strong and weak convergence theorems of the iteration scheme (1.2).

Wang[4] studied the following iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \qquad n \ge 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0, 1), $T_1, T_2 : K \to E$ are two asymptotically nonexpansive nonself mappings, and proved strong and weak convergence theorems of the iteration scheme (1.3). Guo and Guo[5] completed the weak convergence theorems of the iteration scheme (1.3).