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Strong Convergence for a Countable Family of Total Quasi-ϕ-asymptotically Nonexpansive Nonself Mappings in Banach Space

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Abstract: The purpose of this article is to introduce a class of total quasi-ϕ-asymptotically nonexpansive nonself mappings. Strong convergence theorems for common fixed points of a countable family of total quasi-ϕ-asymptotically nonexpansive mappings are established in the framework of Banach spaces based on modified Halpern and Mann-type iteration algorithm. The main results presented in this article extend and improve the corresponding results of many authors.

Key words: strong convergence, total quasi-ϕ-asymptotically nonexpansive nonself, generalized projection

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1 Introduction and Preliminaries

Throughout this article we assume that E is a real Banach space with norm $\|\cdot\|$, E^* is the dual space of E, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the duality pairing between E and E^*, C is a nonempty closed convex subset of E, $\mathbf{N}$ and $\mathbf{R}^+$ denote the set of natural numbers and the set of nonnegative real numbers, respectively. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2;\ \|f^*\| = \|x\|,\ x \in E\}$$

is called the normalized duality mapping. Let $T : C \to C$ be a nonlinear mapping, and $F(T)$ denotes the set of fixed points of mapping T.

A subset C of E is said to be retract if there exists a continuous mapping $P : E \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach

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space is a retraction. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all $z \in R(P)$, the range of P. A mapping $P : E \to C$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from E to C.

In this paper, we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. We use $\phi : E \times E \to \mathbf{R}^+$ to denote the Lyapunov function, which is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \qquad x, y \in E.$$

It is obvious that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \qquad x, y \in E, \tag{1.1}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x, y) + (1-\lambda)\phi(x, z),$$
$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \qquad x, y, z \in E. \tag{1.2}$$

Following Alber[1], the generalized projection $\Pi_C x : E \to C$ is defined by

$$\Pi_C x = \arg\inf_{y\in C} \phi(y, x), \qquad x \in E.$$

Lemma 1.1[1] *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$ *for all* $x \in C$, $y \in E$;

(ii) *If* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x$ *if and only if* $\langle z - y, Jx - Jz\rangle \ge 0$ *for all* $y \in C$;

(iii) *For any* $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Lemma 1.2[2] *Let E be a uniformly convex and smooth Banach space, and $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then* $\|x_n - y_n\| \to 0$.

Recently, many researchers have focused on studying the convergence of iterative scheme for quasi-ϕ-asymptotically nonexspansive mappings and total quasi-ϕ-asymptotically nonexspansive mappings. Related works can be found in [3–10]. The quasi-ϕ-nonexspansive, quasi-ϕ-asymptotically nonexspansive and total quasi-ϕ-asymptotically nonexspansive mappings are defined as:

Definition 1.1 *A mapping $T : C \to C$ is said to be quasi-ϕ-nonexpansive, if $F(T) \ne \emptyset$ and $\phi(u, Tx) \le \phi(u, x)$ holds for all $x \in C$, $u \in F(T)$.*

A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \ne \emptyset$, and there exists a sequence $\{k_n\} \subset [1, +\infty]$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \le k_n\phi(p, x)$ holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbf{N}$.

A mapping $T : C \to C$ is said to be total quasi-ϕ-asymptotically nonexpansive, if $F(T) \ne \emptyset$, and there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ with $\psi(0) = 0$ such that

$$\phi(p, T^n x) \le \phi(p, x) + \mu_n\psi(\phi(p, x)) + \nu_n$$

holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbf{N}$.