COMMUNICATIONS IN MATHEMATICAL RESEARCH
30(1)(2014), 11–22

Open Loop Saddle Point on Linear Quadratic Stochastic Differential Games

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Communicated by Li Yong

Abstract: In this paper, we deal with one kind of two-player zero-sum linear quadratic stochastic differential game problem. We give the existence of an open loop saddle point if and only if the lower and upper values exist.

Key words: stochastic differential game, saddle point, open loop strategy

2000 MR subject classification: 91A23

Document code: A

Article ID: 1674-5647(2014)01-0011-12

1 Introduction

In this paper, we consider the two-player zero-sum linear quadratic stochastic differential games on a finite horizon. The fundamental theory of differential games was given in 1965 by [1]. Pontryagin's Maximum Principle (see [2]) and Bellman's Dynamic Programming (see [3]) are applied to games. Bensoussan[4], Bensoussan and Friedman[5] studied stochastic differential games. It is well known that the existence of open loop saddle points guarantees the existence of the value of the differential games; the existence and equivalence of the lower and upper values guarantee the existence of the value of the differential games. These statements can be found, for instance, in [6–8].

Zhang[9] considered the two-person linear quadratic differential games and showed that the value of the game exists if and only if both the upper and lower values exist. The same outcomes were proved by Delfour[10] by using another way. Specially, Mou and Yong[11] discussed two-person zero-sum linear quadratic stochastic differential games in Hilbert spaces. The stochastic form of this problem is studied in this paper and we can achieve the same outcomes: No need of equivalence of the lower and upper values, we can prove the existence of the saddle point if and only if the lower and upper values exist. Due to stochastic op-

Received date: Jan. 4, 2011.

Foundation item: The Young Research Foundation (201201130) of Jilin Provincial Science & Technology Department, and Research Foundation (2011LG17) of Changchun University of Technology.

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timal control (see [4], [12]) is concerned, in the present paper we use the Peng's stochastic maximum principle (see [12]) to gain the adjoint equation of this stochastic state system.

This paper is organized as follows: Section 2 provides the basic framework. Some results of payoff function are discussed in Section 3. The main outcomes are characterized in Section 4, where we prove the existence of the saddle point by the existence of lower and upper values in this differential game.

2 Statement of the Problem

Let Ω be a bounded smooth domain in $\mathbf{R}^n$, $(\Omega,\ \mathcal{F},\ P)$ be a probability space with filtration $\mathcal{F}^t$, and $W(\ \cdot\ )$ be an $\mathbf{R}^n$-valued standard Wiener process. We assume that

$$\mathcal{F}^t = \sigma\{W(s);\ 0 \le s \le t\}.$$

Let $\boldsymbol{x}$ be a solution of the following stochastic differential equation:

$$\begin{cases} \mathrm{d}\boldsymbol{x}(t) = (\boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}_1(t)\boldsymbol{u}(t) + \boldsymbol{B}_2(t)\boldsymbol{v}(t))\mathrm{d}t \\ \qquad\qquad +(\boldsymbol{D}(t)\boldsymbol{x}(t) + \boldsymbol{C}_1(t)\boldsymbol{u}(t) + \boldsymbol{C}_2(t)\boldsymbol{v}(t))\mathrm{d}w_t, \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{2.1}$$

where $\boldsymbol{x}_0$ is the initial state at time $t = 0$. We call that $\boldsymbol{u}(t) \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^m)$, $m \ge 1$, is the strategy of the first player if, $\boldsymbol{u}(\ \cdot\ )$ is an $\mathcal{F}^t$-adapted process with values in U (a nonempty subset of $\mathbf{R}^m$ (control domain)) such that

$$E\bigg(\int_0^T |\boldsymbol{u}(t)|^2 \mathrm{d}t\bigg) < \infty,$$

and $\boldsymbol{v}(t) \in \mathcal{L}^2(0,\ T;\ \mathbf{R}^k)$, $k \ge 1$, is the strategy of the second player.

For any choice of controls $\boldsymbol{u}$, $\boldsymbol{v}$, we have the following payoff function:

$$C_{\boldsymbol{x}_0}(\boldsymbol{u},\ \boldsymbol{v}) = \frac{1}{2}E\Big(\boldsymbol{F}\boldsymbol{x}(T)\cdot\boldsymbol{x}(T) + \int_0^T \boldsymbol{Q}(t)\boldsymbol{x}(t)\cdot\boldsymbol{x}(t) + |\boldsymbol{u}(t)|^2 - |\boldsymbol{v}(t)|^2\mathrm{d}t\Big). \tag{2.2}$$

We assume that $\boldsymbol{F}$ is an $n \times n$ matrix, and $\boldsymbol{A}(t)$, $\boldsymbol{B}_1(t)$, $\boldsymbol{B}_2(t)$, $\boldsymbol{C}_1(t)$, $\boldsymbol{C}_2(t)$, $\boldsymbol{D}(t)$ and $\boldsymbol{Q}(t)$ are matrix functions of appropriate order that are measurable and bounded a.e. in $[0,T]$. Moreover, $\boldsymbol{F}$ and $\boldsymbol{Q}(t)$ are symmetrical. We write $\boldsymbol{A}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{D}$ and $\boldsymbol{Q}$ instead of $\boldsymbol{A}(t)$, $\boldsymbol{B}_1(t)$, $\boldsymbol{B}_2(t)$, $\boldsymbol{C}_1(t)$, $\boldsymbol{C}_2(t)$, $\boldsymbol{D}(t)$ and $\boldsymbol{Q}(t)$ throughout this paper and use the above assumptions. $T > 0$ is a given final time. $|\boldsymbol{x}|$ and $\boldsymbol{x}\cdot\boldsymbol{y}$ are the usual norm and inner product, respectively.

The more general quadratic structure involving cross terms and different quadratic weights $\boldsymbol{N}_1\boldsymbol{u}\cdot\boldsymbol{u}$ and $\boldsymbol{N}_2\boldsymbol{v}\cdot\boldsymbol{v}$ on $\boldsymbol{u}$ and $\boldsymbol{v}$ can be simplified to our model (see [10]).

Definition 2.1 *The game is said to achieve its open loop lower value if*

$$v^-(\boldsymbol{x}_0) = \sup_{\boldsymbol{v}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^k)}\ \inf_{\boldsymbol{u}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^m)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v})$$

is finite and is said to achieve its open loop upper value if

$$v^+(\boldsymbol{x}_0) = \inf_{\boldsymbol{u}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^m)}\ \sup_{\boldsymbol{v}(t)\in\mathcal{L}^2(0,T;\mathbf{R}^k)} C_{\boldsymbol{x}_0}(\boldsymbol{u},\boldsymbol{v})$$

is finite.