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Variational Approach to Scattering by Inhomogeneous Layers Above Rough Surfaces

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Abstract: In this paper, we study, via variational methods, the problem of scattering of time harmonic acoustic waves by unbounded inhomogeneous layers above a sound soft rough surface. We first propose a variational formulation and exploit it as a theoretical tool to prove the well-posedness of this problem when the media is non-absorbing for arbitrary wave number and obtain an estimate about the solution, which exhibit explicitly dependence of bound on the wave number and on the geometry of the domain. Then, based on the non-absorbing results, we show that the variational problem remains uniquely solvable when the layer is absorbing by means of a priori estimate of the solution. Finally, we consider the finite element approximation of the problem and give an error estimate.

Key words: Helmholtz equation, rough surface, scattering problem

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1 Introduction

This paper is concerned with the study of a boundary value problem for the Helmholtz equation modeling scattering of time harmonic waves by a layer above an unbounded rough surface on which the field vanishes. Such problems arise frequently in practical applications, such as in modeling outdoor noise propagation or sonar measurements in oceanography.

In this paper we focus on a particular, typical problem of the class, which models acoustic waves scattering by inhomogeneous layer above a sound soft unbounded rough surface. Since the unboundedness of the scattering object present a major challenge, mathematical methods to solve such scattering problem are often difficult to develop. Nevertheless, a variety of different methods and techniques have been introduced during the last years. Most of them were concerned with Drichlet boundary value problems for the Helmholtz equation with

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constant real wave number (see [1–7]).

The idea of our argument is inspired by [5–6], in which a Rellich identity was used to prove the estimates for solutions of the Helmholtz equation posed on unbounded domains. Though the results and methods were closest to those of Chandler[5,7], who studied the similar problem tackled in those papers, and considered the homogeneous media only for non-absorbing case in [5], and obtained the well posed results just only for the wave number which is small enough in [7].

The main results of this paper are as follows: In Section 2, we introduce the boundary value problem considered in this paper. Then we propose the variational formulation, which is used as a theoretical tool to analyze the well-posedness of the problem. In Section 3, we consider the non-absorbing case. We first establish a Rellich-type identity, from which the inf-sup condition of the sesquilinear form follows. Then the existence and uniqueness of the solution to variational problem can be deduced by application of the generalized Lax-Milgram theory of Babuska (see [8]). In Section 4, we turn our interest to the absorbing scatterers, and establish the uniqueness via a priori estimate which also leads to an existence result based on the non-absorbing results. In Section 5, the finite element approximation of the problem is considered. Finally, we analyze the convergence and error estimate.

2 Boundary Value Problem and Variational Formulation

In this section, we first define some notations related to the problem. Then we introduce the boundary value problem and its equivalent variational formulation to be analyzed later. For $x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n$ $(n=2,3)$, let $\tilde{x}=(x_1,x_2,\cdots,x_{n-1})$ so that $x=(\tilde{x},x_n)$. For $H\in\mathbf{R}$, let $U_H:=\{x\mid x_n>H\}$ and $\varGamma_H:=\{x\mid x_n=H\}$. Suppose that D is a connected open set with some constants $f_-<f_+$. Then it holds that

$$U_{f_+}\subset D\subset U_{f_-}, \tag{2.1}$$

and

$$x+s\boldsymbol{e}_n\in D,\qquad s>0,\ x\in D, \tag{2.2}$$

where $\boldsymbol{e}_n$ denotes the unit vector in the direction of x_n. Let $\varGamma=\partial D$ and $S_H:=D\backslash\overline{U}_H$ for some $H\ge f_+$. Moreover, we assume that the wave number k satisfies

$$\begin{cases} 0\le k\le k_+, & x\in D;\\ k(x)=k_0>0, & x\in U_H;\\ \dfrac{\partial k^2(x)}{\partial x_n}\ge 0\ , & x\in S_H. \end{cases} \tag{2.3}$$

Next we introduce the main function spaces in which we set our problem. The Hilbert space V_H is defined by

$$V_H=\{\phi|_{S_H}:\phi\in H_0^1(D)\},\qquad H\ge f_+,$$

on which we impose the wave number dependent scalar product

$$(u,v)_{V_H}:=\int_{S_H}(\nabla u\cdot\nabla\bar{v}+k_0^2u\bar{v})\mathrm{d}x,$$