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The Supersolvable Order of Hyperplanes of an Arrangement

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Abstract: This paper mainly gives a sufficient and necessary condition for an order of hyperplanes of a graphic arrangement being supersolvable. In addition, we give the relations between the set of supersolvable orders of hyperplanes and the set of quadratic orders of hyperplanes for a supersolvable arrangement.

Key words: quadratic arrangement, graphic arrangement, supersolvable order of hyperplane, quadratic order of hyperplane, supersolvable order of vertices

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1 Introduction

As we know, a central question in the theory of hyperplane arrangements is the problem of expressing topological invariants of the complement space in terms of combinatorics. Therefore, the study of arrangements with rational $K[\pi, 1]$ complements has a relatively long history (see [1–3]). Assume that $M(\mathcal{A})$ is the complement of the arrangement $\mathcal{A}$. Then a necessary condition for $M(\mathcal{A})$ being rational $K[\pi, 1]$ is that the Orlik-Solomon algebra $A(\mathcal{A})$ is quadratic (see [4–5]). Therefore, the study of quadratic Orlik-Solomon algebras attracts more and more attentions (see [6]). Pearson[7] gave the definition of a quadratic arrangement, and showed that if $\mathcal{A}$ is a quadratic arrangement, then its Orlik-Solomon algebra $A(\mathcal{A})$ is quadratic. Hence, it is important to study quadratic arrangements. It was pointed out in [8] that if $\mathcal{A}$ is a supersolvable arrangement, then $\mathcal{A}$ is a quadratic arrangement under some supersolvable order of hyperplanes. A natural question is whether there exists a good method to characterize all the supersolvable orders of hyperplanes for a supersolvable arrangement, and what is the relation between the supersolvable order of

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hyperplanes and the order which makes $\mathcal{A}$ quadratic. We give the answers of these two questions in this paper.

Firstly, this paper gives a sufficient and necessary condition of an order of hyperplanes of a graphic arrangement being supersolvable. Secondly, we prove that the set of the supersolvable orders of hyperplanes is strictly contained in the set of quadratic orders of hyperplanes for any supersolvable arrangement.

We assume that $\mathcal{A}$ is a central arrangement in this paper.

2 Basic Notions

Let $\mathbb{K}$ be a field and V be a vector space of dimension n on $\mathbb{K}$. A hyperplane H in V is an affine subspace of dimension $(n-1)$. A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes in V. Most often we take $\mathbb{K} = \mathbf{R}$. We simply use the term arrangement for a hyperplane arrangement. We call $\mathcal{A}$ central if $\bigcap\limits_{H\in\mathcal{A}} H = T \neq \emptyset$. Let $L = L(\mathcal{A})$ be the set of nonempty intersections of hyperplanes in $\mathcal{A}$. Define $X \leq Y$ in L if $X \supseteq Y$. In other words, L is partially ordered by reverse inclusion. We call L the intersection poset of $\mathcal{A}$. Let $X \vee Y$ be the least upper bound of X and Y, and $X \wedge Y$ be the greatest lower bound of X and Y. For any $X \in L$, the rank of X is given by $r(X) = \operatorname{codim}(X)$. A pair $(X, Y) \in L \times L$ is called a modular pair if

$$r(X) + r(Y) = r(X \vee Y) + r(X \wedge Y).$$

An element $X \in L$ is called a modular element if (X, Y) is a modular pair for all $Y \in L$. We call $\mathcal{A}$ supersolvable if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_r = T,$$

where $r = r(T)$.

We introduce an arbitrary linear order "$\prec$" on the hyperplanes of $\mathcal{A}$, that is, $H_i \prec H_j$ if $1 \leq i < j \leq |\mathcal{A}|$.

Let $\mathcal{A}$ be a supersolvable arrangement with order $\prec$. We say that the order $\prec$ is a supersolvable order of hyperplanes if there exists a maximal modular chain

$$V = X_0 < X_1 < \cdots < X_r = T$$

in L such that

(1) $X_1 = H_1$;

(2) for $1 < i \leq r$, there exists $n_i \geq 2$ such that

$$X_i = \bigcap_{j=1}^{n_i} H_j,$$

and if $\widetilde{H} < X_i$, then

$$\widetilde{H} \in \{H_1, \cdots, H_{n_i}\}.$$

We denote by $\prec_s$ a supersolvable order of hyperplanes.

Let $S = (H_{i_1}, \cdots, H_{i_p})$ be a p tuple of $\mathcal{A}$. We say that an S is dependent if $r(\cap S) < |S|$, otherwise, an S is independent. A p-tuple is a circuit if it is minimally dependent. We call