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Continuous Selections and Fixed Points for ϕ-maps with Their Applications to Section Problems*

Piao Yong-jie, Jin Hai-lan and Piao Guang-ri
(*Department of Mathematics, College of Science, Yanbian University, Yanji, Jilin, 133002*)

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Abstract: The concept of finitely continuous topological space is introduced and the basic properties of the space are given. Several continuous selection theorems and fixed point theorems for ϕ-maps are established, and as applications of the above fixed point theorems, some section problems are discussed. The results generalize and improve many corresponding conclusions.

Key words: FC-space, FC-subset, ϕ-map, continuous selection, fixed point, section problem

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1 Introduction and Preliminaries

Motivated by the well-known works of Horvath[1–2], there have appeared many generalizations of a convex subset of a topological vector space. The most general one seems to be that of finitely continuous topological space or FC-space introduced by Ding[3–4], which extends many topological spaces having generalized convexity structures.

Browder[5–6] showed that any multimap with convex valued and open fibers from a Hausdorff compact space to a convex space has a continuous selection, and used this fact to obtain the famous Fan-Browder fixed point theorem. Later it was generalized and applied by many authors (see, for example, [7–12]).

The concept of ϕ-map was introduced by Hovarth[2]. In recent years, it has been extended by many authors. Several continuous selections of ϕ-maps were obtained and some applications were given (see [12–14]).

In this paper, we introduce the concept of finitely continuous topological space (simply, FC-space), which was given by Ding[3–4], and define the concept of ϕ-map in FC-spaces, and

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obtain two basic properties of FC-spaces. And then, we establish some continuous selection theorems for ϕ-maps and give several fixed point theorems for composite of two ϕ-maps. Finally, we discuss the section problems as applications of the above fixed point theorems.

Definition 1.1 *$(X, D;\ \{\varPhi_N\})$ is called a finitely continuous topological space (simply, FC-space) if D is a nonempty subset of a topological space X, and for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ (where some elements of N may be same), there exists a continuous map $\varPhi_N : \triangle_n \to X$, where $\triangle_n$ is the standard n-simplex in $\mathbf{R}^{n+1}$.*

Definition 1.2 *Let $(X, D;\ \{\varPhi_N\})$ be an FC-space, Y and Z be two subsets of X. Z is called an FC-subset of X relative to Y if for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ (where some elements of N may be same) and each $J = \{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset N \cap Y$, $\varPhi_N(\triangle_k) \subset Z$, where $\triangle_k$ is the face of $\triangle_n$ corresponding to J. If $Z = Y$, then Z is called an FC-subset of X.*

Definition 1.3 *Let X be a nonempty set, and $(Y, D;\ \{\varPhi_N\})$ be an FC-space. $F : X \multimap Y$ is called a ϕ-map if there exists a map $S : X \multimap D$ (which is called the companion map of F) such that*

(i) *for each $x \in X$, $F(x)$ is an FC-subset of Y relative to $S(x)$;*

(ii) *$X = \bigcup\{\mathrm{Int}S^-(y) : y \in D\}$.*

Theorem 1.1 *Let $\{(X_i, D_i;\ \{\varPhi_{N_i}\})\}_{i\in I}$ be a family of FC-spaces. Then $(\prod\limits_{i\in I} X_i,\ \prod\limits_{i\in I} D_i;\ \{\varPhi_N\})$ is also an FC-space, where $\varPhi_N : \triangle_n \to \prod\limits_{i\in I} X_i$ is defined by $\varPhi_N(\alpha) = \prod\limits_{i\in I} \varPhi_{N_i}(\alpha)$ for each $\alpha \in \triangle_n$ with $N \in \langle \prod\limits_{i\in I} D_i\rangle$ and $N_i = \pi_i(N)$.*

Proof. Let

$$N = \{a_0, a_1, \cdots, a_n\} \in \langle \prod_{i\in I} D_i\rangle.$$

Then

$$N_i := \pi_i(N) = \{a_{0,i}, a_{1,i}, \cdots, a_{n,i}\} \in \langle D_i\rangle,$$

and N_i has $n+1$ elements for all $i \in I$. Hence there exists a continuous map $\varPhi_{N_i} : \triangle_n \to X_i$ for each $i \in I$. Let $\varPhi_N : \triangle_n \to \prod\limits_{i\in I} X_i$ be defined by

$$\varPhi_N(\alpha) = \prod_{i\in I} \varPhi_{N_i}(\alpha), \qquad \alpha \in \triangle_n.$$

Clearly, $\varPhi_N$ is continuous. Hence $(\prod\limits_{i\in I} X_i,\ \prod\limits_{i\in I} D_i;\ \{\varPhi_N\})$ is an FC-space.

Theorem 1.2 *Let $\{(X_i, D_i;\ \{\varPhi_{N_i}\})\}_{i\in I}$ be an family of FC-spaces, and Z_i be an FC-subset of X_i relative to Y_i for each $i \in I$. Then $\prod\limits_{i\in I} Z_i$ is an FC-subset of the FC-space $(\prod\limits_{i\in I} X_i,\ \prod\limits_{i\in I} D_i;\ \{\varPhi_N\})$ relative $\prod\limits_{i\in I} Y_i$, where $\varPhi_N = \prod\limits_{i\in I} \varPhi_{N_i}$ is as that in Theorem 1.1.*