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A Quasilinear Parabolic System with Nonlocal Boundary Conditions and Localized Sources*

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Abstract: This paper investigates the properties of solutions to a quasilinear parabolic system with nonlocal boundary conditions and localized sources. Conditions for the existence of global or blow-up solutions are given. Global blow-up property and blow-up rate estimates are also derived.

Key words: global existence, finite time blow-up, localized source, nonlocal boundary, global blow-up, blow-up rate
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1 Introduction

In this paper, we consider the positive classical solutions to the porous medium system with nonlocal boundary conditions and localized sources

$$\begin{cases} u_t = \Delta u^m + av^p(x_0, t), & x \in \Omega,\ t > 0, \\ v_t = \Delta v^n + bu^q(x_0, t), & x \in \Omega,\ t > 0, \\ u(x,t) = \displaystyle\int_\Omega k_1(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ v(x,t) = \displaystyle\int_\Omega k_2(x,y)v(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $m,\ n > 1$, a, b, p, $q > 0$ are constants, Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $k_1(x,y)$, $k_2(x,y) \not\equiv 0$ are nonnegative continuous functions defined for $x \in \partial\Omega$ and $y \in \bar{\Omega}$, while $u_0(x)$, $v_0(x)$ are positive continuous functions and satisfy the compatibility conditions

$$u_0(x) = \int_\Omega k_1(x,y)u_0(y)\mathrm{d}y$$

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and

$$v_0(x) = \int_\Omega k_2(x,y)v_0(y)\mathrm{d}y$$

for $x \in \partial\Omega$.

The properties of solutions to partial differential equations with local boundary conditions have been discussed in [1] and [2]. However, there are some important phenomena formulated into parabolic equations which are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [3]–[5]). In this case, the solution $u(x,t)$ describes entropy per volume of material.

The parabolic problem with nonlocal boundary condition of the type

$$\begin{cases} u_t = \Delta u + g(x,u), & x \in \Omega,\ t > 0, \\ u(x,t) = \displaystyle\int_\Omega k(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

was studied by Friedman[6]. He established the global existence of its solution, and showed that the unique solution tends to 0 monotonically and exponentially as $t \to +\infty$ in the case of

$$g(x,u) = c(x)u$$

with $c(x) \le 0$ and

$$\int_\Omega |k(x,y)|\mathrm{d}y < 1, \qquad x \in \partial\Omega.$$

In 1992, Deng[7] gave the comparison principle and local existence of classical solutions to (1.2) with general $g(x,u)$. For the case

$$g(x,u) = c(x)u,$$

he showed that the solution exists globally and may increase at most exponentially with t under some weaker assumptions than those in [6]. Blow-up results of (1.2) are due to Seo[8]. He investigated (1.2) with

$$g(x,u) = g(u)$$

and gave the blow-up condition of positive solutions by using supersolution and subsolution method. The blow-up rate estimates for the special case

$$g(u) = u^p$$

and

$$g(u) = e^u$$

were also derived.

A more general problem with nonlocal boundary conditions was investigated by Pao[9,10], where the following problem

$$\begin{cases} u_t = Lu + g(x,u), & x \in \Omega,\ t > 0, \\ Bu = \displaystyle\int_\Omega k(x,y)u(y,t)\mathrm{d}y, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.3}$$