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$\boldsymbol{\Psi}$-bounded Solutions for a System of Difference Equations on $\mathbb{Z}$*

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Abstract: In this work we discuss the existence of $\boldsymbol{\Psi}$-bounded solutions for linear difference equations. We present a necessary and sufficient condition for the existence of $\boldsymbol{\Psi}$-bounded solutions for the linear nonhomogeneous difference equation $\boldsymbol{x}(n+1) = \boldsymbol{A}(n)\boldsymbol{x}(n) + \boldsymbol{f}(n)$ for every $\boldsymbol{\Psi}$-bounded sequence $\boldsymbol{f}(n)$.

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1 Introduction

The difference equations play an important role in many scientific fields, such as scientific computing, numerical analysis of ordinary and partial differential equations, control theory and computer science (see [1]–[3] and references therein). The behavior of solutions of difference equations has been paid much more attention by mathematicians and scientists, and the boundedness of solutions is closely related to the investigation of numerical discretization for differential equations (see [1], [2] and [4]). The boundedness of the solution of ordinary differential equations is a very important property. For example, the existence of the bounded solution implies the existence of the almost periodic solution (see [5]).

The problem of boundedness of the solutions for the system of ordinary differential equations

$$\boldsymbol{x}' = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{f}(t)$$

was studied by Coppel[6]. Diamandescu[7],[8],[9] proposes the concept of $\boldsymbol{\Psi}$-boundedness of solutions, which is interesting and useful in some practical cases for differential equations, and presents the existence condition for such solutions. Han[10] defined $\boldsymbol{\Psi}$-boundedness of

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solutions for difference equations

$$\boldsymbol{x}(n+1) = \boldsymbol{A}(n)\boldsymbol{x}(n) + \boldsymbol{f}(n) \tag{1.1}$$

via $\boldsymbol{\Psi}$-bounded sequences and established a necessary and sufficient condition for existence of $\boldsymbol{\Psi}$-bounded solutions for every $\boldsymbol{\Psi}$-summable sequence $\boldsymbol{f}$ on $\mathbb{N}$. Diamandescu[11] gave a necessary and sufficient condition for the existence of $\boldsymbol{\Psi}$-bounded solutions for the nonhomogeneous linear difference equation (1.1) for every $\boldsymbol{\Psi}$-summable sequence $\boldsymbol{f}$ on $\mathbb{Z}$.

The aim of this paper is to give a necessary and sufficient condition for the nonhomogeneous system of ordinary difference equations (1.1) to have at least one $\boldsymbol{\Psi}$-bounded solution on $\mathbb{Z}$ for every $\boldsymbol{\Psi}$-bounded sequence $\boldsymbol{f}$ on $\mathbb{Z}$, where $\mathbb{Z}$ is the integer set.

Let $\mathbb{R}^d$ be the Euclidean d-space, and

$$\boldsymbol{x} = (x_1,\ x_2,\ \cdots,x_d)^T, \qquad \|\boldsymbol{x}\| = \max\{|x_1|,\ \cdots,\ |x_d|\}.$$

For a $d\times d$ matrix $\mathbf{M}$, define the norm

$$\|\mathbf{M}\| = \sup_{\|\boldsymbol{x}\|\le 1}\|\mathbf{M}\boldsymbol{x}\|.$$

Let

$$\Psi_i : \mathbb{Z} \to (0,+\infty), \qquad i = 1,2,\cdots,d,$$

and

$$\boldsymbol{\Psi} = \operatorname{diag}[\Psi_1,\ \Psi_2,\ \cdots,\ \Psi_d].$$

The matrix $\boldsymbol{\Psi}(n)$ is invertible for each $n\in\mathbb{Z}$.

Definition 1.1 *A sequence $\boldsymbol{\varphi}:\mathbb{Z}\to\mathbb{R}^d$ is said to be $\boldsymbol{\Psi}$-bounded on $\mathbb{Z}$ if $\boldsymbol{\Psi}(n)\boldsymbol{\varphi}(n)$ is bounded on $\mathbb{Z}$.*

Definition 1.2 *A sequence $\boldsymbol{\varphi}:\mathbb{Z}\to\mathbb{R}^d$ is said to be $\boldsymbol{\Psi}$-summable on $\mathbb{Z}$ if $\sum\limits_{-\infty}^{+\infty}\boldsymbol{\Psi}(n)\boldsymbol{\varphi}(n)$ is convergent.*

We assume that $\{\boldsymbol{A}(n)\}$ is a bounded matrix sequence and the associated linear difference system is

$$\boldsymbol{y}(n+1) = \boldsymbol{A}(n)\boldsymbol{y}(n). \tag{1.2}$$

Let $\boldsymbol{Y}$ be the fundamental matrix of (1.2) with $\boldsymbol{Y}(0) = \boldsymbol{I}_d$, the identity $d\times d$ matrix.

Let the vector space $\mathbb{R}^d$ be represented as a direct sum of three subspace X_-, X_0, X_+ such that a solution $y(n)$ of (1.2) is $\boldsymbol{\Psi}$-bounded on $\mathbb{Z}$ if and only if $\boldsymbol{y}(0)\in X_0$, and $\boldsymbol{\Psi}$-bounded on $\mathbb{Z}_+$ if and only if $\boldsymbol{y}(0)\in X_-\bigoplus X_0$, and $\boldsymbol{\Psi}$-bounded on $\mathbb{Z}_-$ if and only if $\boldsymbol{y}(0)\in X_+\bigoplus X_0$. Also, let P_-, P_0, P_+ denote corresponding projection onto X_-, X_0, X_+, respectively.

Lemma 1.1 *Let $\boldsymbol{Y}(n)$ be an invertible matrix on $\mathbb{Z}_+$ and P be a projection. If there exists a sequence $\phi:\mathbb{Z}_+\to(0,+\infty)$ and a positive constant M such that*

$$\sum_{m=0}^{n}\phi(m)|\boldsymbol{\Psi}(n)\boldsymbol{Y}(n)P\boldsymbol{Y}^{-1}(m+1)\boldsymbol{\Psi}^{-1}(m)|\le M, \qquad n\ge 0$$

and

$$\sum_{m=0}^{\infty}\phi(m) = +\infty,$$