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Asymptotic Property of Approximation to $x^\alpha \mathrm{sgn} x$ by Newman Type Operators*

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Abstract: The approximation of $|x|$ by rational functions is a classical rational problem. This paper deals with the rational approximation of the function $x^\alpha \mathrm{sgn} x$, which equals $|x|$ if $\alpha = 1$. We construct a Newman type operator $r_n(x)$ and show

$$\max_{|x|\le 1}\{|x^\alpha \mathrm{sgn} x - r_n(x)|\} \sim Cn^{-\frac{\alpha}{2}} \mathrm{e}^{-\sqrt{2n\alpha}},$$

where C is a constant depending on α.

Key words: rational approximation, asymptotic property, Newman type operator

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1 Introduction

In 1964, Newman[1] demonstrated that rational approximation to the function $|x|$ is much more favorable in contrast to polynomial approximation, more exactly, $|x|$ can be approximated uniformly by rational functions with a rate of $\mathrm{e}^{-c\sqrt{n}}$. Newman's proof is based on the following construction.

Let

$$p(X;x) = \prod_{k=1}^{n-1}(x + a^k),$$

where

$$X = \{\pm a^k\}_{k=1}^{n-1} \cup \{0\}, \qquad a = \exp\{-n^{-\frac{1}{2}}\}, \quad n = 1, 2, \cdots$$

Then, the rational approximant $R_n(X;x)$ is defined by

$$R_n(X;x) = x\frac{p(X;x) - p(X;-x)}{p(X;x) + p(X;-x)}. \tag{1.1}$$

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Making use of the above rational interpolating function, Newman got a well-known inequality as follows:

$$\frac{1}{2}\mathrm{e}^{-9\sqrt{n}} \le \max_{|x|\le 1}\{||x| - R_n(X;x)|\} \le 3\mathrm{e}^{-\sqrt{n}}, \qquad n \ge 5,\ x \in [-1,1]. \tag{1.2}$$

After Newman's result published, a great deal of researches on rational approximation to $|x|$ occurred. Much of them focused on sharpening the estimation for the error of the best rational approximation. And many others concentrated on sharpening the estimations of the approximation rates for interpolating rational functions with given kinds of nodes, for example, Newman's nodes X, equidistant nodes and Tchebuchev nodes, etc. Especially, what is the exact approximation rate to $|x|$ by Newman's interpolating rational operators? Xie and Zhou[2] investigated this problem and gave an asymptotic formula as follows:

$$\mathrm{e}^{\sqrt{n}}\sqrt{n}\max_{|x|\le 1}\{||x| - R_n(x)|\} = A + O\left(\frac{1}{\sqrt{n}}\right), \qquad n \to \infty, \tag{1.3}$$

where

$$A = \max_{0\le x<\infty}\left\{\frac{x}{1+\mathrm{e}^x}\right\} = 0.27846\cdots$$

It is a classical result of Bernstein that the sequence of Lagrange interpolation polynomials to $|x|$ at equally spaced nodes in $[-1,\ 1]$ diverges everywhere, except at zero and the end-points. In 2000, Revers[3] proved the same conclusion for $|x|^\alpha$ $(0 \le \alpha \le 1)$. In 2006, Su and Xu[4] promoted this result to $|x|^\alpha$ $(2 < \alpha < 4)$. There was also a good many of the studies on rational approximation to $|x|^\alpha$, but much of them did not give concrete approximating functions and asymptotic formulae.

In this paper, we focus the approximated function to

$$f(x) = x^\alpha \mathrm{sgn} x, \qquad x \in [-1,\ 1],$$

where $\alpha = \dfrac{q}{p} > 0$ is a reduced fraction and p is an odd number. Clearly, $f(x) = |x|$ if $\alpha = 1$, and $f(x) = |x|^\alpha$ if q is an odd number. We modify Newman's construction and apply this to the approximation of the function mentioned above. Our new system of nodes on $[-1,\ 1]$ is

$$X_\alpha = \{\pm x_k = \varepsilon^{\frac{k}{n}}\}_{k=1}^{n-1} \cup \{0\},$$

where

$$\varepsilon = \mathrm{e}^{-\sqrt{\frac{2n}{\alpha}}}.$$

Let

$$p(X_\alpha; x) = \prod_{k=1}^{n-1}(x + \varepsilon^{\frac{k}{n}}),$$

and the Newman type operator

$$r_n(X_\alpha; x) = x^\alpha\,\frac{p(X_\alpha; x) - p(X_\alpha; -x)}{p(X_\alpha; x) + p(X_\alpha; -x)}. \tag{1.4}$$

In Section 2, making use of Lemmas 2.1 and 2.2 we find the upper bound of $|x^\alpha \mathrm{sgn} x - r_n(X_\alpha; x)|$. Especially, the inequality in Lemma 2.2

$$\prod_{k=1}^{n-1}\frac{1-\varepsilon^{\frac{k}{n}}}{1+\varepsilon^{\frac{k}{n}}} \le \exp\left\{-\frac{10}{9}\sqrt{2\alpha n} + \frac{8}{3} + \frac{28}{27}\cdot\frac{\alpha}{\mathrm{e}}\right\}$$