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KAM Type-Theorem for Lower Dimensional Tori in Random Hamiltonian Systems*

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Abstract: In this paper, we study the persistence of lower dimensional tori for random Hamiltonian systems, which shows that majority of the unperturbed tori persist as Cantor fragments of lower dimensional ones under small perturbation. Using this result, we can describe the stability of the non-autonomous dynamic systems.
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1 Introduction

We consider the persistence of lower dimensional tori for a family of random real analytic Hamiltonian systems of the parameterized action-angle form

$$H = e + \langle \omega, y\rangle + \frac{1}{2}\langle z, Mz\rangle + \varepsilon P(x, y, z, \theta_t), \tag{1.1}$$

where $(x, y, z) \in \mathbf{T}^d \times \mathbf{R}^d \times \mathbf{R}^{2m}$ varies in a complex neighborhood $D(r, s) = \{(x, y, z) : |\mathrm{Im} x| < r, |y| < s^2, z < s\}$ of $\mathbf{T}^d \times \{0\} \times \{0\}$, $\omega \in \mathcal{O}$ (a bounded closed region in $\mathbf{R}^d$), ε is a small parameter, $\theta_t : \Omega \subset \mathbf{R}^d \to \Omega$, $t \in \mathbf{R}^1_+$, is a continuous stationary stochastic processes with $\theta_0 = \mathrm{id}$, and $(\Omega, P, \mathcal{F})$ is a stochastic basis. Hereafter, all θ_t dependence function are of class C^{l_0} for some $l_0 \geq d$, and P is a small perturbation.

This kind of systems describes dynamics of harmonic oscillator under perturbations such as white noise, or under some effects of some noise θ_t which are neither periodic, quasi-periodic nor almost periodic.

With the symplectic form

$$\sum_{i=1}^{d} \mathrm{d}x_i \wedge \mathrm{d}y_i + \sum_{j=1}^{m} \mathrm{d}z_j \wedge \mathrm{d}z_{d+j},$$

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the equation of motion of (1.1) reads

$$\begin{cases} \dot{x} = \omega + \varepsilon \dfrac{\partial P}{\partial y}, \\ \dot{y} = -\varepsilon \dfrac{\partial P}{\partial x}, \\ \dot{z} = JMz + J\dfrac{\partial P}{\partial z}, \end{cases}$$

where $M(\theta_t)$ is a $2m \times 2m$ real symmetric matrix for each $\theta_t \in \Omega$, and J is the standard $2m \times 2m$ symplectic matrix. Hence the associated unperturbed motion of (1.1) is simply described by the equation

$$\begin{cases} \dot{x} = \omega, \\ \dot{y} = 0, \\ \dot{z} = JMz, \end{cases}$$

which implies that the unperturbed system admits a family of invariant tori $T_\omega = \mathbf{T}^d \times \{0\} \times \{0\}$ parameterized by the frequency vectors $\omega \in \mathcal{O}$.

Similar to the classical KAM theorem (see [1]–[3]), Melnikov[4],[5] posed the persistence problem of lower dimensional tori in the deterministic Hamiltonian systems, which concludes that under some appropriate non-degenerate and non-resonance conditions, there exits a Cantor set $\mathcal{O}_* \subset \mathcal{O}$, such that those lower dimensional invariant d-tori with the frequencies $\omega \in \mathcal{O}_*$ will persist as ε sufficiently small; moreover, in the sense of Lebesgue measure $\mathcal{O}_* \to \mathcal{O}$, as $\varepsilon \to 0$. Some achievements on Melnikov persistence problem can be found in [6]–[20].

However, what happens to the Melnikov persistence for random or non-periodical perturbed systems (1.1)? In this paper, we are concern with this problem. We prove that for most of frequencies $\omega \in \Omega$, there exists a set of Cantor set $\Omega_\gamma \subset \Omega$ such that the associated unperturbed lower dimensional invariant torus T_ω, $\omega \in \Omega_\gamma$, persists as a set of Cantor fragments of the invariant torus with the"random frequency" close to $\omega(\theta_t)$ for the perturbed system (1.1), provided ε is sufficiently small.

The persistence of lower dimensional tori problem can describe the stability of non-autonom-ous systems. Different from previous, we need not to assume that the perturbation P is periodic or not. Applying the results, we know that there is a Cantor set Ω_γ, such that when $\theta_t \in \Omega_\gamma$, the lower dimensional invariant tours of unperturbed system persists, provided ε is sufficiently small.

The paper is organized as follows. In Section 2, we state our theorem for a general random Hamiltonian system and the corollary A of non-autonomous systems. Then, a parameter-depended iterative scheme is described in Section 3 for one cycle. In Section 4, we derive the proof of our result by deriving an iteration lemma and giving measure estimates.