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A WENO-Based Stochastic Galerkin Scheme for Ideal MHD Equations with Random Inputs

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Abstract. In this paper, we investigate the ideal magnetohydrodynamic (MHD) equations with random inputs based on generalized polynomial chaos (gPC) stochastic Galerkin approximation. A special treatment with symmetrization is carried out for the gPC stochastic Galerkin method so that the resulting deterministic gPC Galerkin system is provably symmetric hyperbolic in the spatially one-dimensional case. We discretize the hyperbolic gPC Galerkin system with a high-order path-conservative finite volume weighted essentially non-oscillatory scheme in space and a third-order total variation diminishing Runge-Kutta method in time. The method is also extended to two spatial dimensions via the operator splitting technique. Several numerical examples are provided to illustrate the accuracy and effectiveness of the numerical scheme.

AMS subject classifications: 65M60, 65M08, 35L60, 76W05

Key words: Uncertainty quantification, ideal magnetohydrodynamics, generalized polynomial chaos, stochastic Galerkin, symmetric hyperbolic, finite volume WENO method.

1 Introduction

Magnetohydrodynamic (MHD) equations play an important role in many fields such as astrophysics, space physics, plasma physics, etc. Especially the ideal MHD equations are widely used to model the dynamics of electrically-conducting fluid in the presence of magnetic fields. In practical applications, the ideal MHD equations may often contain uncertainties, which can come from various model coefficients, initial or boundary

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conditions, etc; see, for example, [22,25,47] for some discussions on the importance of uncertainty quantification (UQ) for fluid dynamics or MHD simulations. Random variables are thus introduced to represent these uncertainties, rendering an otherwise deterministic ideal MHD equations as stochastic. UQ has received much attention, become a crucial part of scientific computing, and is very important for risk/sensitivity analysis and evaluating the reliability of the simulations.

In recent years, a few numerical techniques have been developed to quantify the uncertainty in MHD equations. In [21,22], the authors presented the recent results on the design, analysis and implementation of efficient statistical sampling methods of the Monte Carlo (MC) and multi-level Monte Carlo (MLMC) finite volume method for solving uncertain hyperbolic systems including applications to the MHD equations. The authors in [36] compared the MC, MLMC and stochastic collocation methods for two plasma fluid model. In [25], by modeling the velocity and the resistivity as random variables in the MHD model, the authors quantified the effects of uncertainty on the induced magnetic field and then developed stochastic expressions for these quantities and investigate their impact within a finite element discretization, where mean and variance data are obtained through Monte Carlo simulation.

In this paper, we investigate the MHD equations with uncertainties based on the stochastic Galerkin method of generalized polynomial chaos (gPC) approximation. The original PC method [15] was inspired by the Wiener chaos expansion which uses Hermite polynomials to represent Gaussian stochastic processes. Later the approach was extended to gPC [49] where general orthogonal polynomials are adopted for more general random processes. The gPC method is mathematically robust and is numerically easy to implement with either stochastic Galerkin (SG) methods or stochastic collocation methods. We refer the readers to the book [46] for more details of the gPC methods.

Although the gPC-SG method has been successfully applied to a large variety of problems, its application to the MHD equations with uncertainty is very challenging, due to the complicated mathematical structure and the nonlinear hyperbolic nature of the equations. One of the difficulties is caused by the fact that for general hyperbolic systems, the resulting gPC-SG system may not be always hyperbolic, see e.g. [10] for shallow water equations and Euler equations. This also happens to the ideal MHD equations. The lack of hyperbolicity may lead to the ill-posedness of the initial or boundary problem and the instability of the numerical simulations. For scalar conservation laws or linear hyperbolic systems with uncertainty only on initial or boundary conditions, the resulting gPC-SG systems are still hyperbolic, see e.g. [4, 17, 19, 30, 40]. For nonlinear hyperbolic systems, several efforts were made to obtain the well-behaved gPC-SG system [5,6,10,11,13,14,27,33,44]. The early work in [10] proved that the resulting gPC-SG system for Euler equations is hyperbolic, by reformulating the system in a symmetrically hyperbolic form in terms of the entropy variables. Another approach was proposed for the Euler equations using the Roe variables in [27]. A class of operator splitting based SG methods was developed for the Euler equations in [5] and for the Saint-Venant system in [6]. Recently, a new SG framework was proposed in [44] for first-order quasilinear hy-