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Grad-div Stabilized Finite Element Schemes for the Fluid-Fluid Interaction Model

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Abstract. In this work, two fully discrete grad-div stabilized finite element schemes for the fluid-fluid interaction model are considered. The first scheme is standard grad-div stabilized scheme, and the other one is modular grad-div stabilized scheme which adds to Euler backward scheme an update step and does not increase computational time for increasing stabilized parameters. Moreover, stability and error estimates of these schemes are given. Finally, computational tests are provided to verify both the numerical theory and efficiency of the presented schemes.

AMS subject classifications: 65M15, 65M60

Key words: Fluid-fluid interaction model, grad-div stabilized scheme, stability, error estimate.

1 Introduction

In this paper, we consider the following fluid-fluid interaction model. Given kinematic viscosities $\nu_i>0$ $(i=1,2)$, friction coefficient $\kappa\in\mathbb{R}$, f_i: $[0,T]\to H^1(\Omega_i)^2$ and $u_{i,0}(x)\in H^1(\Omega_i)^2$, find $u_i: [0,T]\times\Omega_i\to\mathbb{R}^2$ and $p_i: [0,T]\times\Omega_i\to\mathbb{R}$ satisfying:

$$\begin{aligned}
&u_{i,t}-\nu_i\Delta u_i+u_i\cdot\nabla u_i+\nabla p_i=f_i && \text{in } \Omega_i,\\
&-\nu_i n_i\cdot\nabla u_i\cdot\tau=\kappa|u_i-u_j|(u_i-u_j)\cdot\tau && \text{on } I,\ i,j=1,2,\ i\neq j,\\
&u_i\cdot n_i=0 && \text{on } I,\\
&\nabla\cdot u_i=0 && \text{in } \Omega_i,\\
&u_i(0,x)=u_{i,0}(x) && \text{in } \Omega_i,\\
&u_i=0 && \text{on } \Gamma_i=\partial\Omega_i\backslash I,
\end{aligned}\tag{1.1}$$

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where $|\cdot|$ represents the Euclidean norm and u_i, p_i and f_i represent the velocity, pressure and body force on Ω_i, respectively. The bounded domain Ω of the model (1.1) is comprised by two subdomains Ω_1 and Ω_2, which are coupled across their shared interface I. Both subdomains are the subsets of $\mathbb{R}^2$ and have outward unit normal vectors n_i. Besides, τ is any vector such that $\tau \cdot n_i = 0$. Note that the nonlinear interface conditions on I are applied in this fluid-fluid model and the interface I is used to model a line segment.

As is known, many important applications require an accurate solution of multi-domain and multi-physics coupling of one fluid with another fluid [3, 4, 9, 38]. In fact, the model (1.1) can be seen as one of them arises in many important scientific, engineering and industrial applications, such as heterogeneous of blood flow [14] and atmosphere-ocean interaction [27]. Although this model can reduce the dynamic core of the atmosphere-ocean problem to its simplest form, it still retains some inevitable difficulties of the problem. Besides, in terms of numerical methods, constructing stable and efficient numerical schemes for the model (1.1) is challenging, since it involves the coupling of the pressure, the incompressible conditions, the nonlinearity and the complicated coupled system with some nonlinear interface conditions. Therefore, much effort has been throwing to the development of efficient decoupled numerical methods for investigating the model (1.1).

By using geometric averaging method to deal with the nonlinear interface conditions, several stable schemes have been constructed for the fluid-fluid interaction model in [10, 34]. Bresch and Koko [6] have presented a numerical simulation of the considered model by using an operator-splitting method and optimization-based nonoverlapping domain decomposition methods, which solves one coupled degenerated Stokes problem and one uncoupled linear advection-diffusion problem. Based on implicit-explicit scheme for the nonlinear interface conditions, Connors et al. [12] have presented a decoupled time stepping method which is conditionally stable proved by Zhang et al. [37], although it is the simplest and most natural decoupled method [11]. Besides, Connors [7] has proposed a statistical turbulence model for ensemble calculations with two fluids coupled across a flat interface. Aggul et al. [1] have developed a predictor-corrector-type method that is an unconditionally stable scheme with second-order time accuracy.

Unlike the above methods for the fluid-fluid interaction model (1.1), the grad-div stabilized finite element schemes are considered in this paper, which can penalize for lack of mass conservation and improve solution accuracy by reducing the effect of the pressure on the velocity error. The grad-div stabilization is firstly introduced in [16] and is a simple, useful and popular technique for incompressible flow problems. Hence, there is an abundance of literature regarding this method [23, 26, 29, 31–33, 36]. The optimal error estimates of velocity and pressure for the evolutionary Oseen equations are obtained by Frutos et al. [17], where they used inf-sup stable mixed finite elements in a Galerkin method and gained the error bounds with constants independent of ν. Further, they have extended the previous work to the Navier–Stokes equations with high Reynolds number [18]. A new sparse grad-div stabilization has been proposed by Linke et al. [25] as a more efficient alternative for the standard grad-div stabilization. A combination