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A Second-Order Scheme with Nonuniform Time Steps for a Linear Reaction-Subdiffusion Problem

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Abstract. It is reasonable to assume that a discrete convolution structure dominates the local truncation error of any numerical Caputo formula because the fractional time derivative and its discrete approximation have the same convolutional form. We suggest an error convolution structure (ECS) analysis for a class of interpolation-type approximations to the Caputo fractional derivative. Our assumptions permit the use of adaptive time steps, such as is appropriate for accurately resolving the initial singularity of the solution and also certain complex behavior away from the initial time. The ECS analysis of numerical approximations has two advantages: (i) to localize (and simplify) the analysis of the approximation error of a discrete convolution formula on general nonuniform time grids; and (ii) to reveal the error distribution information in the long-time integration via the global consistency error. The core result in this paper is an ECS bound and a global consistency analysis of the nonuniform Alikhanov approximation, which is constructed at an offset point by using linear and quadratic polynomial interpolation. Using this result, we derive a sharp L^2-norm error estimate of a second-order Crank-Nicolson-like scheme for linear reaction-subdiffusion problems. An example is presented to show the sharpness of our analysis.

AMS subject classifications: 65M06, 35B65

Key words: Caputo fractional derivative, nonuniform time mesh, error convolution structure, global consistency error, stability and convergence.

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1 Introduction

The time-fractional diffusion equation provides a valuable tool for modeling complex systems such as glassy and disordered media [11]. This paper builds on our recent results [12, 14, 15, 17] for the nonuniform mesh technique applied to the time discretization of the following reaction-subdiffusion problem [6] in a bounded domain $\Omega\subset\mathbb{R}^d$ ($d=1$, 2, 3),

$$\begin{aligned} &\mathcal{D}_t^\alpha u-\triangle u=\kappa u+f(\boldsymbol{x},t) && \text{for } \boldsymbol{x}\in\Omega \text{ and } 0<t<T,\\ &u=u_0(\boldsymbol{x}) && \text{for } \boldsymbol{x}\in\Omega \text{ when } t=0, \end{aligned} \tag{1.1}$$

subject to the homogeneous Dirichlet boundary condition $u=0$ on $\partial\Omega$. Here, the reaction coefficient κ is a real constant, and $\mathcal{D}_t^\alpha={}_0^{\mathrm{C}}\mathcal{D}_t^\alpha$ denotes the Caputo fractional derivative of order α ($0<\alpha<1$) with respect to time t, that is,

$$(\mathcal{D}_t^\alpha v)(t):=\int_0^t \omega_{1-\alpha}(t-s)v'(s)\,\mathrm{d}s \quad \text{for } t>0, \quad \text{where } \omega_\beta(t):=t^{\beta-1}/\Gamma(\beta).$$

1.1 Initial singularity and the nonuniform time meshes technique

In developing numerical methods for solving the subdiffusion problem (1.1), an important issue to be considered is that the solution u is typically less regular than in the case of a classical parabolic PDE (as the limiting case $\alpha\to 1$). Sakamoto and Yamamoto [23] showed that if the initial data $u^0\in H^2(\Omega)\cap H_0^1(\Omega)$, then the unique solution $u\in C\big([0,T];H^2(\Omega)\cap H_0^1(\Omega)\big)$, with $\mathcal{D}_t^\alpha u\in C\big([0,T];L^2(\Omega)\big)$ and $\partial_t u\in L^2(\Omega)$. However, $\|\partial_t u(t)\|_{L^2(\Omega)}\le C_u t^{\alpha-1}$ for $0<t\le T$, where the constant $C_u>0$ is independent of t but may depend on T. In fact, u can only be a smooth function of t if the initial data and source term satisfy some restrictive compatibility conditions [24].

The focus of this paper is on a second-order time discretization of (1.1). The spatial discretization is of less interest: we apply the standard Galerkin finite element method based on the weak form of the fractional PDE,

$$\langle\mathcal{D}_t^\alpha u,v\rangle+\langle\nabla u,\nabla v\rangle=\kappa\langle u,v\rangle+\langle f(t),v\rangle \quad \text{for all } v\in H_0^1(\Omega) \text{ and for } 0<t\le T,$$

where $\langle u,v\rangle$ denotes the usual inner product in $L_2(\Omega)$. Thus, we construct the usual space of continuous, piecewise-linear functions with respect to a partition of Ω into subintervals (in 1D), triangles (in 2D) or tetrahedrons (in 3D) with the maximum diameter h, and let X_h denote the subspace of functions satisfying the homogeneous Dirichlet boundary condition. In the usual way, the (semidiscrete) Galerkin finite element solution $u_h:[0,T]\to X_h$ is then defined by requiring that

$$\langle\mathcal{D}_t^\alpha u_h,\chi\rangle+\langle\nabla u_h,\nabla\chi\rangle=\kappa\langle u_h,\chi\rangle+\langle f(t),\chi\rangle \quad \text{for all } \chi\in X_h \text{ and for } 0<t\le T,$$

with $u_h(0)=u_{0h}\approx u_0$ for a suitable $u_{0h}\in X_h$.