JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **34**, No. 2, pp. 116-128

doi: 10.4208/jpde.v34.n2.2
June 2021

Trudinger-Moser Type Inequality Under Lorentz-Sobolev Norms Constraint

ZHU Maochun* and ZHENG Yifeng

School of Mathematical Sciences, Institute of Applied System Analysis, Jiangsu University, Zhenjiang 212013, China.

Received 5 November 2020; Accepted 5 January 2021

Abstract. In this paper, we are concerned with a sharp fractional Trudinger-Moser type inequality in bounded intervals of $\mathbb{R}$ under the Lorentz-Sobolev norms constraint. For any $1<q<\infty$ and $\beta\le(\sqrt{\pi})^{q'}\equiv\beta_q$, $q'=\frac{q}{q-1}$, we obtain

$$\sup_{u\in\tilde{H}^{1/2,2}(I),\left\|(-\Delta)^{1/4}u\right\|_{2,q}\le 1}\int_I e^{\beta|u(x)|^{q'}}\mathrm{d}x\le c_0|I|,$$

and β_q is optimal in the sense that

$$\sup_{u\in\tilde{H}^{1/2,2}(I),\left\|(-\Delta)^{1/4}u\right\|_{2,q}\le 1}\int_I e^{\beta|u(x)|^{q'}}\mathrm{d}x=+\infty,$$

for any $\beta>\beta_q$. Furthermore, when q is even, we obtain

$$\sup_{u\in\tilde{H}^{1/2,2}(I),\left\|(-\Delta)^{1/4}u\right\|_{2,q}\le 1}\int_I h(u)e^{\beta_q|u(x)|^{q'}}\mathrm{d}x=+\infty,$$

for any function $h:[0,\infty)\to[0,\infty)$ with $\lim_{t\to\infty}h(t)=\infty$. As for the key tools of proof, we use Green functions for fractional Laplace operators and the rearrangement of a convolution to the rearrangement of the convoluted functions.

AMS Subject Classifications: 46E30, 46E35, 26D15

Chinese Library Classifications: O178

Key Words: Trudinger-Moser inequality; Lorentz-Sobolev space; bounded intervals.

---

*Corresponding author. *Email addresses:* `zhumaochun2006@126.com` (M. C. Zhu), `2590519973@qq.com` (Y. F. Zheng)

1 Introduction

Let $\Omega \subset \mathbb{R}^n (n\geq 2)$ denote a bounded domain in $\mathbb{R}^n$. Then the Sobolev embedding theorem assures that $W_0^{1,n}(\Omega) \subset L^q(\Omega)$ for any $q \in [1,+\infty)$, but easy examples show that $W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega)$. One knows from the works by Yudovich [1], Pohozaev [2] and Trudinger [3] that $W_0^{1,n}(\Omega)$ embeds into the Orlicz space $L_\varphi(\Omega)$, with N-function $\varphi=\exp\left(|t|^{\frac{n}{n-1}}-1\right)$. This embedding was made more precise by J. Moser[4], and obtained the following inequality (now is called *Trudinger-Moser inequality*)

$$\sup_{u\in W_0^{1,n}(\Omega),\|\nabla u\|_n^n\leq 1}\int_\Omega e^{\alpha u^{n/(n-1)}}\mathrm{d}x<\infty, \quad \text{iff} \;\; \alpha\leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}, \tag{1.1}$$

where ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$. This result has led to many related results: extension of Trudinger-Moser inequality to unbounded domains, see [5–9], an analog of (1.1) for higher order derivatives is established by Adams [10], extension of Trudinger-Moser inequality in Lorentz spaces, see [11–13]. For other important extensions can see [14–18], and the reference therein.

Recently, S. Lula, A. Maalaoui and L. Martinazzi [19] established the Trudinger-Moser type inequality in one dimension, which can be sated as the following inequality:

$$\sup_{u\in \tilde{H}^{1/p,p}(I)||(-\Delta)^{\frac{1}{2p}}u||_{L^p(I)}\leq 1}\int_I e^{\alpha_p|u|^{\frac{p}{p-1}}}\mathrm{d}x = c_p|I|, \quad I\subset\mathbb{R},$$

where $\tilde{H}^{1/p,p}$ is the fractional Sobolev Spaces and the constant

$$\alpha_p=\frac{1}{2}\left[2\cos\left(\frac{\pi}{2p}\right)\Gamma\left(\frac{1}{p}\right)\right]^{p'}\left(p'=\frac{p}{p-1}\right)$$

is optimal.

In this work, we are interested in the Trudinger-Moser type inequality in one dimension space under the Lorentz-Sobolev norms constraint. In order to state the main results of the paper we first focus our attention on the fractional Sobolev spaces and introduce some relevant function spaces.

Let us consider the space

$$L_s(\mathbb{R}^n)=\left\{u\in L^1_{loc}(\mathbb{R}^n):\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+s}}\mathrm{d}x<\infty\right\},$$

where $s\in(0,1)$. For functions $\varphi\in L_s(\mathbb{R}^n)$, we define the fractional Laplacian $(-\Delta)^{s/2}\varphi$ as follows.

$$(-\Delta)^{s/2}\varphi:=\mathcal{F}^{-1}(|\xi|^s\mathcal{F}\varphi(\xi)),$$