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A Class of Robust Low Dissipation Nested Multi-Resolution WENO Schemes for Solving Hyperbolic Conservation Laws

Zhenming Wang[1], Jun Zhu[2], Yuchen Yang[1], Linlin Tian[1] and Ning Zhao[1,*]

[1] *College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, China*
[2] *College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, China*

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Abstract. In this paper, an efficient class of finite difference nested multi-resolution weighted essentially non-oscillatory (WENO) schemes with increasingly higher order of accuracy is presented for solving hyperbolic conservation laws on structured meshes. The crucial idea is originated from [44, 46]. We only use the information defined on a series of nested unequal-sized central spatial stencils in high-order spatial approximations. These new nested multi-resolution WENO schemes use the same large central stencil as the same order classical WENO schemes [3,21] without increasing the number of the total spatial stencils, could obtain the optimal order of accuracy in smooth regions, and could control spurious oscillations nearby strong shocks or contact discontinuities by gradually degrading from ninth-order to seventh-order, fifth-order, third-order or ultimately to the first-order accuracy. Associated linear weights can be set as any positive numbers on condition that their summation is one. Therefore, these new WENO schemes are simple to construct and can be easily implemented to arbitrarily high-order accuracy in multi-dimensions. Some benchmark examples are illustrated to show the good performance of these new nested multi-resolution WENO schemes, especially in terms of robustness and low numerical dissipation.

AMS subject classifications: 65M60, 35L65

Key words: Nested multi-resolution scheme, finite difference WENO scheme, unequal-sized central spatial stencil, high-order accuracy, hyperbolic conservation laws.

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*Corresponding author.
Emails: wangzhenming@nuaa.edu.cn (Z. M. Wang), zhujun@nuaa.edu.cn (J. Zhu), ycyang0320@nuaa.edu.cn (Y. C. Yang), lins_tian@nuaa.edu.cn (L. L. Tian), zhaoam@nuaa.edu.cn (N. Zhao)

1 Introduction

The hyperbolic conservation laws play an important role in many fields of engineering, such as computational fluid dynamics (CFD) [1, 8, 11, 12, 20], magneto-hydrodynamics (MHD) [2], traffic flow [39], and many others [30, 31]. For such problems, the solutions may develop strong discontinuities even though their initial conditions are smooth enough, and therefore their numerical simulations are more challenging. Consequently, the high-order numerical methods for solving hyperbolic conservation laws attract many attentions in recent decades and show great capability in computing flows with multi-scale structures [37]. So the main objective of this paper is to design new high-order spatial reconstruction procedures together with a classical third-order Runge-Kutta time discretization method [33] to obtain associated nested multi-resolution weighted essentially non-oscillatory (WENO) schemes on structured meshes. Although this new spatial reconstruction methodology can be applied to any high-order reconstruction procedures in multi-dimensions, we will only design third-order, fifth-order, seventh-order, and ninth-order cases in this paper as examples for simplicity.

Among many high-order numerical schemes, the weighted essentially non-oscillatory (WENO) schemes [31] are very popular for solving hyperbolic conservation laws. The reason is that WENO schemes have many advantages, including their capability of achieving high-order accuracy in smooth regions, maintaining stable property, and keeping non-oscillatory property and sharp discontinuity transitions nearby strong shocks or contact discontinuities [31]. It is well known that the WENO schemes are designed based on the successful application of essentially non-oscillatory (ENO) schemes in [18, 33, 34]. In 1994, the first one-dimensional third-order finite volume WENO scheme was introduced by Liu, Osher, and Chan in their pioneering paper [26]. In 1996, Jiang and Shu provided a general framework to construct third-order and fifth-order finite difference WENO schemes [21], which are more efficient in simulating multi-dimensional problems. Both ENO and WENO schemes use the idea of adaptive spatial stencils in spatial reconstruction procedures based on the local smoothness of the numerical solution to automatically achieve high-order accuracy in smooth regions and keep essentially non-oscillatory property near strong discontinuities [30]. The main difference between them is ENO schemes use only one smoothest stencil in all candidate stencils when performing spatial reconstructions, while WENO schemes use a convex combination of all candidate stencils, each of them is assigned a nonlinear weight. By doing so, these WENO schemes improve ENO schemes in many aspects [30]. Based on their pioneering work of [21], many scholars have developed various studies on WENO schemes in many fields. For example, the high accuracy ADER-WENO schemes were designed for hydrodynamics and divergence-free magnetohydrodynamics problems [2], a class of WENO schemes with adaptive order were studied by Balsara [1, 3], the WENO-M scheme [19] and the WENO-Z schemes [4, 7, 10] were designed for achieving optimal order near critical points, a finite volume WENO type solver for steady-state Euler equations [20], the finite volume WENO schemes on unstructured meshes [27], a class of central WENO