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Efficient and Stable Schemes for the Magnetohydrodynamic Potential Model

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Abstract. In this paper, we consider the numerical approximations of a magnetohydrodynamic potential model that was developed in [15]. Several decoupled, linear, unconditionally energy stable schemes are developed by combining some subtle implicit-explicit treatments for nonlinear coupling terms and the projection-type method for the Navier-Stokes equations. The divergence-free condition for the magnetic field is preserved in the fully-discrete level. We further establish the well-posedness and unconditional energy stabilities of the proposed schemes and present a series of numerical examples in 3D, including accuracy/stability tests, benchmark simulations of driven cavity flow and hydromagnetic Kelvin-Helmholtz instability.

AMS subject classifications: 65M60
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1 Introduction

Magnetohydrodynamic (MHD) system describes the dynamic behaviors of an electrically conducting fluid under the influence of an external magnetic field. It occurs in geophysics, astrophysics, fusion reactor blankets, and confinement for controlled thermonuclear fusion, see [11, 26, 30]. The model involves multi-physics, thus the governing equations couple the Navier-Stokes equations for hydrodynamics and Maxwell's equations for electromagnetism. The two equations are coupled by the Lorentz force, which governs the effect of a magnetic field on fluid flow, and the appearance of the fluid velocity in Ohm's law, which accounts for the influence of hydrodynamics on the electric current. Concerning the corresponding extensive theoretical/numerical studies including the modeling and PDE analysis of the MHD system, we refer to [1–3, 9, 13, 14, 17–19, 21, 24, 29, 32, 33, 39] and the references therein.

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The magnetic field $\boldsymbol{B}$ in MHD models usually satisfies the divergence-free condition ($\nabla\cdot\boldsymbol{B}=0$) that is a precise physical law in electro-magnetics which implies that there is no source of the magnetic field in the domain. It is very important to preserve this condition in the discrete level since it has been shown that even tiny perturbations to this condition may lead to large errors in numerical simulations, see [5,8,22,38]. Recently, a novel, so-called MHD potential model is developed in [15], in which a magnetic potential vector field $\boldsymbol{A}$ is introduced and the magnetic field is then defined as its rotation, i.e., $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. In this way, the numerical solution of the magnetic field $\boldsymbol{B}_h$ can be recovered automatically by $\boldsymbol{B}_h=\nabla\times\boldsymbol{A}_h$ which in turn ensures the exact divergence-free condition. Meanwhile, to solve the model, two numerical schemes (one linear and one nonlinear) based on the Crank-Nicolson methods were developed in [15]. However, the algorithms are coupled type which implies that the fluid velocity, the magnetic potential, and the pressure are all coupled together at each time step.

Hence, in this paper, we aim to develop some more efficient numerical schemes to solve the MHD potential model developed in [15]. We are particularly interested in designing energy stable schemes, in the sense that the discrete energy dissipation laws may hold. In the meantime, while keeping the energy stable feature, we prefer to develop schemes that are easy-to-implement which is referred to linear and decoupled in comparison with its counterparts: nonlinear and coupled. To this end, the main challenging issue that is needed to overcome is to develop suitable temporal discretizations for a large number of nonlinear and coupling terms, including (i) the coupling between the velocity and pressure in the fluid momentum equation; (ii) the nonlinear coupling between the magnetic potential and the fluid velocity field through the convection; and (iii) the nonlinear coupling between the fluid velocity and the magnetic potential through the Lorentz force. We expect to construct a time discretization scheme which (a) is unconditionally stable; (b) satisfies a discrete energy law; and (c) leads to decoupled equations to solve at each time step. This is by no means an easy task due to the highly nonlinear coupling nature appears in the MHD potential model.

The rest of the paper is organized as follows. In Section 2, we present the MHD potential model and derive the associated energy dissipation law. In Section 3, we present a first-order scheme and two second-order schemes and prove their well-posedness and unconditional energy stabilities. In Section 4, a series of 3D numerical examples are implemented including accuracy/stability tests, benchmark simulations of driven cavity flow and hydromagnetic Kelvin-Helmholtz instability to demonstrate the stability and accuracy of the schemes. Finally, some concluding remarks are given in Section 5.

2 The potential MHD model and its energy law

In this section, we present the magnetic potential MHD model and demonstrate its energy dissipation law. Throughout this paper, we consider the incompressible MHD problem in a bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ with the boundary $\partial\Omega$. The magnetic poten-